Adaptive wavelet-based multifractal model applied to the effective bandwidth estimation of network traffic flows

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Abstract: The authors investigate effective bandwidth estimation and Quality of Service (QoS) aware bandwidth provisioning for multifractal network traffic flows. They develop a novel adaptive wavelet-based multifractal model (AWMM) by using properties of the wavelet coefficients of multifractal cascade processes. The proposed AWMM has real-time updating capability and proves to be efficient in capturing multifractal network traffic characteristics. In addition, the authors derive an analytical expression for the effective bandwidth estimation of AWMM traffic flows, capable of being used to meet desired byte loss probabilities. Finally, they present an online effective bandwidth estimation algorithm that is incorporated into an adaptive bandwidth provisioning scheme and comparatively evaluated against some other bandwidth allocation methods.

1 Introduction

The effective bandwidth of a traffic source has been widely accepted as one of the most appropriate traffic link parameters for call admission control and resource allocation within computer network environments [1]. The effective bandwidth of a traffic flow is the rate that can be used to estimate the link capacity required to support the flow at the required Quality of Service (QoS), given the amount of available buffer space [2]. For example, variable bit rate connections can transport multiplexed traffic with a rate less than the total sum of the peak rates of all individually involved connections as soon as a small loss or delay is allowed. In this case, the effective bandwidth can be viewed as a way of characterising the resource requirements of variable rate connections.

Nowadays, IP technology is widely used by existing service networks. However, most of these IP networks implement solely ‘best-effort’ policies that are well suited for data transmission rather than time-sensitive applications. In order to support time-sensitive services and variable rate transmissions, the concept of QoS must be incorporated in or implemented over IP networks. The first tentative approach was IntServ (integrated service) that proposes resource reservations along entire connection paths across the network [3]. Because of the lack of scalability, this QoS model cannot be satisfactorily applied to Internet cores whenever the number of individual flows grows high.

As an alternative, DiffServ (differentiated services) was designed to solve the drawback of non-scalability by aggregating data flows. Instead of individualised treatment for each traffic trace, data packets are assigned to classes that provide distinguished QoS conditioning. DiffServ can be implemented together with a multiprotocol label switching (MPLS) architecture. MPLS incorporates traffic engineering through explicit routing by establishing label switching paths (LSPs) and using forwarding equivalent class that divide traffic into distinct aggregated flows according to service models. In this sense, again, the effective bandwidth can be extremely useful in establishing the service requirements of these flows.

Several approaches have been proposed in the literature to determine the effective bandwidths of different traffic
models, including Markovian [2], short-range- [2] and long-
range-dependent traffic [4, 5]. For long-range-dependent
traffic, the fractional Brownian motion (fBm) is a popular
self-similar model whose effective bandwidth was derived
in [4]. Also, the effective bandwidths of traffic models,
both self-similar and heavy-tailed with stable distributions,
were established in [5]. The popularity of fBm has been
the consequence of the self-similar (fractal) nature of the
Internet traffic demonstrated in previous work [6]. The
self-similar property as well as the long-range dependence
of Gaussian and non-Gaussian network traffic have
been analysed through wavelets in literature, mainly for
completed traces, i.e. requiring an off-line processing time
[7, 8].

Evidences of the existence of multifractal behaviour of
traffic flows in computer networks have contributed to
deeper traffic understanding and modelling [9]. Through
multifractal interpretation, the explanation of traffic
behaviour in timescales smaller than some hundreds of
milliseconds is comprehensible. Hence, multiplicative
cascades have been proposed to model network traffic
processes at small timescales in order to capture their
multifractal characteristics [6]. Unfortunately, in spite of
their interesting and useful approaches, cascade models in
general have numerous parameters that need to be
estimated in order to match the models to completed traffic
traces. This large number of parameters places a heavy
burden on real-time estimation.

In this work, we propose an adaptive wavelet-based
multifractal model (AWMM) that offers online traffic
modelling characterised through a few input parameters. In
the AWMM, we apply wavelets to adaptively compute
some parameters, considering a multiplicative cascade-based
traffic modelling. As a consequence, besides being adequate
for real-time modelling, the AWMM presents similar
modelling performance to that of ‘off-line’ multiplicative
cascades.

An important network management issue is to precisely
determine bandwidths capable of maintaining QoS
requirements in a dynamic environment. Therefore because
dynamically changing traffic environments, adaptive
effective bandwidth computing has become essential. In
this regard, we derive an analytical expression explicitly
estimating effective bandwidths for multifractal cascade
processes. Moreover, having the online updating
characteristics of our adaptive wavelet-based model, we also
propose in this work an adaptive effective bandwidth
estimation algorithm. The novel aspect of this algorithm is
that it is efficient and adaptive, i.e. the algorithm can be
run in real time to estimate bandwidth requirements, thus
allowing effective bandwidth provisioning in QoS-aware
networks. Some obvious advantages of this solution include
that only a few traffic sample values needs to be stored and
effective bandwidths can be computed online based on the
proposed adaptive multifractal model. We envision a wide
use of this network performance tool, for example, in
adaptive resizing of high-speed LSP in MPLS networks
[10], adaptive dimensioning of customer pipes in virtual
private networks [11, 12], and so on.

The organisation of this paper is as follows. In Section 2,
we review some multifractal modelling concepts. In Section
3, we present the AWMM and its synthesis procedure. In
Section 4, the proposed AWMM model is validated via a
variety of statistical and queuing tests and simulations. In
Section 5, we derive an analytical formula for the
estimation of effective bandwidth based on the multifractal
cascade multipliers. In Section 6, this mathematical
expression is used to build an adaptive effective bandwidth
estimation algorithm for multifractal traffic processes as
well as a bandwidth provisioning scheme. Finally in
Section 7, we conclude this study.

2 Multifractal traffic characteristics

Multifractal processes are defined by a scaling law for
moments of the processes’ increments over finite time
intervals. This means the traffic has complex and inherently
strong dependence structures, appearing very bursty with
similar looking burstiness over a wide range of timescales
[13]. Traffic flows with such properties make the network
performance much worse than those of Gaussian and
short-range-dependent traffic types [14]. It is well known
that the increment process of the cumulative traffic load is
generally strongly Gaussian at large scales, whereas non-
Gaussian at small scales [15]. Now, let us formally
introduce the concept of a multifractal process.

Definition 1: A stochastic process \( X(t) \) is called multifractal
if it satisfies

\[
E(|X(t)|^q) = c(q)t^{\tau(q)+1}
\]

for \( t \in T \) and \( q \in Q \), where \( T \) and \( Q \) are intervals on the real
line, and \( \tau(q) \) and \( c(q) \) are functions of domain \( Q \), known as
the scaling factor and the moment factor of the multifractal
process, respectively. Moreover, we assume that \( T \) and \( Q \)
have positive lengths, and that \( 0 \in T, [0,1] \subset Q \). If \( \tau(q) \)
is linear in \( q \), the process is called monofractal; otherwise, it
is multifractal. For self-similar processes with Hurst
parameter \( H \), it can be shown that \( \tau(q) = qH - 1 \) and
\( c(q) = E(|X(1)|^q) \).

The simplest multifractal, the binomial measure, can be
generated on the compact interval \([0, 1]\) through an
iterative procedure called multiplicative cascade. Let \( m_0 \)
and \( m_1 \) be two positive numbers adding up to 1. At stage \( j = 0 \),
we start the construction with the uniform probability
measure \( \mu_0 \) on \([0, 1]\). In the step \( j = 1 \), the measure \( \mu_j \)
uniformly spreads mass equal to \( m_0 \) on the subinterval \([0, 1/2]\)
and mass \( m_1 \) on \([1/2, 1]\). This process is iterated for
\( j \)-levels, and at each stage the total measure is preserved.
If the multipliers used have the same fixed value \(m_0\) then the multiplicative cascade is deterministic [16]. Allowing the cascade multipliers to be random variables, we obtain a stochastic multiplicative cascade. Denoted by \(A_{j,k}\), these multipliers are chosen to be independent random variables with an arbitrary probability distribution function \(f_{j}(x)\).

In multifractal modelling, the wavelet transform has an important role. Discrete wavelet transforms are used for multiscale representation of a process \(X(t)\) through the wavelet and scaling functions \(\psi(t)\) and \(\phi(t)\) as the following

\[
X(t) = \sum_{k} U_{j,k} \phi_{j,k}(t) + \sum_{j=0}^{\infty} \sum_{k} W_{j,k} \varphi_{j,k}(t) \tag{2}
\]

where \(W_{j,k}\) and \(U_{j,k}\) are, respectively, the wavelet and the scaling coefficients given by, respectively

\[
W_{j,k} = \int X(t) \varphi_{j,k}(t) dt \tag{3}
\]

\[
U_{j,k} = \int X(t) \phi_{j,k}(t) dt \tag{4}
\]

Multiplicative cascades in the wavelet domain are capable of characterising a network traffic process \(X(t)\) (e.g. byte/s) by computing its corresponding wavelet coefficients \(W_{j,k}\) as [9, 17]

\[
W_{j,k} = U_{j,k} A_{j,k} \tag{5}
\]

where \(A_{j,k}\) are the cascade multipliers whose values are in the range \([-1, 1]\). Furthermore, some additional conditions are frequently assumed: the multipliers \(A_{j,k}\) are statistically independent and identically distributed (i.i.d) within each scale, independent of \(U_{j,k}\) and symmetric at the origin. A traffic model can capture multifractal characteristics by choosing the multipliers \(A_{j,k}\) in order to control the wavelet coefficient energy \(E(W_{j,k}^2)\). Thus, the following relations are established [9]

\[
\frac{E(W_{j-1,k}^2)}{E(W_{j,k}^2)} = \frac{2p_j + 1}{p_{j-1} + 1} \tag{6}
\]

and

\[
(2p_0 + 1)E(W_{0,0}^2) = E(U_{0,0}^2) \tag{7}
\]

From (6) and (7), it can be seen that the parameter \(p_j\) can be used to capture the decay of the wavelet energy in scale, where \(U_{0,0}\) is the scaling coefficient at the coarsest scale. However, this modelling procedure, called multifractal wavelet model (MWM), requires the determination of a large number of model parameters [6]. This drawback of the MWM makes it unsustainable for online traffic characterisation, since it requires the knowledge of all traffic process samples.

Notice also that the multifractal properties of a real traffic data trace are characterised by its corresponding scaling function \(\tau(q)\) and moment factor function \(\psi(q)\) as described by (1). Therefore a multifractal model is expected to capture these two multifractal properties, which can be fulfilled by the pairwise product of a cascade process \(\mu(t)\) with an arbitrary probability distribution function \(\Delta t\), where \(\Delta t\) represents a time interval in the stage \(j\) of the cascade [18].

The scaling function \(\tau(q)\) can be accurately modelled by choosing the cascade multipliers \(A_{j,k}\) as a symmetric beta random variable with Beta\((\alpha, \alpha)\) distribution for \(\alpha > 0\) and the random variable \(Y\) as a lognormal process whose moment function is \(E(Y^p) = e^{\gamma/2} e^{\gamma^2/2}\), defined by the parameters \(\rho\) and \(\gamma\). Under these assumptions, the scaling function \(\tau(0) = \tau(q) + 1\) and the moment factor \(\psi(q)\) can be, respectively, written as the following [19]

\[
\tau(0) = \log_2 \left( \frac{\Gamma(2\alpha)\Gamma(2\alpha + q)}{\Gamma(2\alpha)\Gamma(\alpha + q)} \right) \tag{8}
\]

and

\[
\psi(q) = e^{\gamma^2/2} \tag{9}
\]

where \(\Gamma(.)\) is the Gamma function.

### 3 Adaptive wavelet-based multifractal model

In this section, we develop a traffic model based on some properties of the Haar wavelet and multiplicative cascades. This multifractal model, namely adaptive wavelet-based model, has some advantages over some existing models in terms of few input parameters and online updating capability.

Our proposal consists of estimating the second moment of the Haar wavelet coefficient of the pairwise product of a cascade and i.i.d. samples of a positive random variable and then generating the model multipliers \(A_{j,k}\) in order to capture the wavelet coefficient energy decay \(E(W_{j,k}^2)\). That is, we introduce a model with other novel characteristic; the model is able to simultaneously capture the functions \(\tau(q)\) and \(\psi(q)\) of multifractal processes and also the wavelet coefficient energy decay. To this end, we derive explicit equations that allow direct computation of the variance of the aggregated traffic process and the second moment of the wavelet coefficients at any time instance.

Let \(X(t)\) be a discrete time process corresponding to the network traffic volume per unit time interval and \(\psi(t)\) be...
the Haar wavelet function given by [17]

$$\phi(t) = \begin{cases} 1 & t \in [0, 1/2) \\ -1 & t \in [1/2, 1) \\ 0 & t \not\in [0, 1) \end{cases}$$  \(10\)

When the representation of a process is done in the Haar wavelet domain, the scaling coefficient \(U_{j,k}\) can be recursively computed as [20]

$$U_{j,2k} = 2^{-1/2}(U_{j-1,k} + W_{j-1,k})$$  \(11\)

$$U_{j,2k+1} = 2^{-1/2}(U_{j-1,k} - W_{j-1,k})$$  \(12\)

In this case, the scaling coefficient \(U_{j,k}\) represents the local mean of the process at different scales and time shifts. The shift \(k_j\) of scaling coefficients is related to the shift of one of its two direct descendants \(k_{j+1}\) in a dyadic cascade as the following [9]

$$k_{j+1} = 2k_j + k_j$$  \(13\)

where \(k_j^l = 0\) corresponds to the left-hand side descendent and \(k_j^r = 1\) to the right-hand side descendent. Two general relations for the coefficients can be stated as follows

$$U_{j,h} = 2^{-j/2}U_{0,0} \sum_{i=0}^{j-1} [1 + (-1)^k A_{j,h}]$$  \(14\)

$$W_{j,h} = 2^{-j/2}A_{j,h} U_{0,0} \sum_{i=0}^{j-1} [1 + (-1)^k A_{j,h}]$$  \(15\)

where \(U_{0,0}\) is assumed to be a Gaussian random variable.

It can be easily demonstrated that the discrete time traffic process \(X(k)\) is related to the scaling coefficients \(U_{j,k}\) at the finest scale, scaled by a factor, i.e.

$$X(k) = 2^{-j/2}U_{j,k}$$  \(16\)

Assuming that \(X(k)\) is a pairwise product process, through Haar wavelet properties, we can derive an analytical expression for the second moment \(E(W_j^2)\) of the wavelet coefficients that can be adaptively computed. We will use this analytical expression to generate the multipliers \(A_{j,k}\) so that the model captures the wavelet energy decay via (6).

The following two lemmas enunciate the variance of the aggregated process \(\text{var}[X^m]\) and the second moment of the wavelet coefficients \(E(W_j^2)\) that will help us in building the AWMM model synthesis procedure.

**Lemma 1:** Let \(X(k)\) be a multifractal process whose \(\tau(q)\) and \(\alpha(q)\) functions are, respectively, given by (8) and (9) and the aggregated process \(X^m\) of \(X(k)\) defined as

$$X^m(i) = \frac{1}{m} \sum_{k=-m+1}^{m} X(k)$$  \(17\)

where \(i = 1, 2, \ldots, L, m = 1, 2, \ldots, 2^N\) and \(L = 2^N/m\). The variance of the aggregated process \(\text{var}[X^m]\) for \(m = 2^j\), \((j = 1, \ldots, N)\), is determined by the following equation

$$\text{var}[X^m] = 2^{-4j} \left( e^{2^j + y^2} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^j - (e^{2^j + y^2})^j \right)$$  \(18\)

**Proof:** See Appendix 10.1.

**Lemma 2:** Assuming that Haar wavelet transform is applied to the process \(X(k)\) which has the \(\tau(q)\) and \(\alpha(q)\) functions, respectively, given by (8) and (9), then the second moment \(E(W_j^2)\) of the wavelet coefficients can be written as

$$E(W_j^2) = e^{2^j + y^2} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^{2^j} - 2Z_j$$  \(19\)

where

$$Z_j = 2^{j-1} \left[ \sigma^2_{\text{var}(2^j)} - e^{2^j + y^2} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^{2^j} + (e^{2^j + y^2})^{2^{j-1}} \right]$$  \(20\)

Moreover, the mean and variance of the scaling coefficients \(U_{j,k}\) at the coarsest scale \(j = N\) are, respectively, given by

$$E(U_{N,k}) = 2^{-N/2}(e^{p+y^2/2})$$  \(21\)

$$\sigma^2_{U_{N,k}} = e^{2^j + y^2} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^{2^j} + 2Z_N - (2^{-N} e^{2^j + y^2})$$  \(22\)

**Proof:** See Appendix 10.2.

The proposed Lemma 1 and Lemma 2 allow the model parameters to be updated without computing the DWT (discrete wavelet transform) for the entire traffic trace. In other words, we do not need to store all the past traffic data in order to update the model parameters as required by the MWM [9] and other traffic models [6, 20]. This property is especially desirable for real-time applications that demand online updating of the traffic model multipliers. Besides, the knowledge of the parameters \((\alpha, \gamma, \rho)\) is enough for the AWMM traffic synthesis procedure.

In order to make our model fully adaptive, we need to periodically update the triple \((\alpha, \gamma, \rho)\) parameters obtained through the scaling function \(\tau(q)\) and the moment factor \(\alpha(q)\).
A method for estimating these functions based on (1) is as follows: given the process increments $X_1, X_2, \ldots, X_t$, we build the corresponding $m$-level aggregated sequence $X^m$ defined as

$$X^m_k = X_{(k-1)m+1} + X_{(k-1)m+2} + \cdots + X_{km}, \quad k = 1, 2, \ldots$$  \hspace{-14pt} (23)

If the sequence $[X_t]$ has scaling properties, then the absolute moments $E(|X|^q)$ versus $m$ on a log-log plot should be a straight line as follows [19]

$$\log E(|X|^q) = \tau_0(q) \log m + \log \epsilon(q) \quad (24)$$

The slope of this straight line provides an estimate of $\tau_0(q)$ and its interception point on the vertical axis is the numerical value of $\log \epsilon(q)$. The proposed AWMM traffic synthesis algorithm includes a recursive least squares algorithm [21] to adaptively obtain the updated values of $\tau_0(q)$ and $\log \epsilon(q)$ on (24).

In the AWMM traffic synthesis algorithm, after obtaining the values for $\tau_0(q)$ and $\log \epsilon(q)$, the Levenberg–Marquardt algorithm is applied to estimate the parameters $\alpha$, $\rho$, and $\gamma$ of functions $\tau_0(q)$ and $\log \epsilon(q)$ [22]. The Levenberg–Marquardt algorithm allows us to accomplish this estimation through the minimisation of a quadratic function $\phi$ [22]

$$\phi = \frac{1}{2} \sum_{j=1}^{N} r_j(x) \quad (25)$$

where $r_j(x)$ corresponds to the error between the measured function value and the analytical function value using parameter $x$.

The procedure for generating the synthetic traffic traces by the AWMM is outlined below:

**Proposed Algorithm 1:** The AWMM traffic synthesis algorithm

1. Let $p_0 = 1$, $\sigma^2_0 = [0]$, $k = 1, \ldots, 2^j$, $j = 1, 2, \ldots, N$ and $q = 1, \ldots, q_2$.
2. Compute $\sigma^2_q = [\tau_0(q) \log \epsilon(q)]$ for each $q$ value using the following recursive equations [21]

$$p_{q,k} = p_{q,k-1} - \rho_{q,k-1} \sigma_q [1 + \sigma^T_{q,k-1} \sigma_q]^{-1} \sigma^T_{q,k-1} p_{q,k-1} \quad (26)$$

$$\sigma^2_q = \sigma^2_{q-1} - \rho_q [x_q^T \sigma^2_{q-1} x_q - x_q^T \sigma^2_{q-1} x_q] \quad (27)$$

where $y^2 = \log E(|X|^q)$, $\tilde{x}_q = [X_1, X_2, \ldots, X_k]$ and $\sigma_q = [\log 2, \ldots, \log k]$.
3. Estimate the $\alpha$ parameter of $\tau_0$ using the Levenberg–Marquardt algorithm according to the following updating rule [22]

$$\alpha_{i+1} = \alpha_i - (H_{e_{\alpha}} + \eta \text{diag}(H))^{-1} \nabla \phi(\alpha_i) \quad (28)$$

where $H_{e_{\alpha}}$ is the Hessian matrix $(H_{e_{\alpha}} = \nabla^2 \phi(\alpha_i))$ of the quadratic error function $\phi$ and $\eta$ is a control parameter at the $i$th iteration of the Levenberg–Marquardt algorithm.

4. Apply again the Levenberg–Marquardt algorithm to estimate the $\rho$ and $\gamma$ parameters of the $\epsilon(q)$ function.

5. Set scale $j = 1$ corresponding to the aggregation fraction equal to $2^j = 2$.

6. Compute the variance of the aggregated process $\text{var}[X^m]$, $\{m = 2^j\}$, through (18)

$$\text{var}[X^m] = 2^{-j} \left( e^{\rho+2\gamma} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^j - (e^{2\rho+\gamma}) \right)$$

7. Compute $Z_j$ by using (20)

$$Z_j = 2^{j-1} \left[ \text{var}[X^{2^j}] - e^{2\rho+2\gamma} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^{2^{-j}} + (e^{2\rho+\gamma})e^{-2^{-j}} \right]$$

8. Compute the second moment of the wavelet coefficients $E(W_j^2)$ of the traffic trace $X(k)$ through (19)

$$E(W_j^2) = e^{2\rho+2\gamma} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^{2^{-j}} - 2Z_j$$

9. Increment $j$ by 1. If $j = N$ (desired maximum number of cascade stages), compute the mean and variance of the scaling coefficients at the coarsest scale using (21) and (22); otherwise, go to step 6.

10. Calculate $\rho_s$ using (6).

11. Here starts a new recursive procedure having the $\rho_s$ values. Set $j = 0$ and compute the scaling coefficient $U_{0,0}$ at the coarsest scale.

12. Generate the random variables $A_{j,k}$ for $k = 1, \ldots, 2^j$ and compute the wavelet coefficients $V_{j,k}$ at scale $j$ using (5).

13. Compute the scaling coefficients at scale $j + 1$, $U_{j+1,2k}$ and $U_{j+1,2k+1}$ for $k = 1, \ldots, 2^j$ using (11) and (12).


15. The generated traffic trace will be $X(k) = 2^{-j/2} U_{j,k}$, $k = 1, \ldots, 2^N$.  

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**IET Commun.,** 2009, Vol. 3, Iss. 6, pp. 906–919
4 Traffic modelling performance

In order to compare the self-similar and multifractal statistical characteristics of the proposed model to those of the real traffic, as well as to those of the MWM model, some statistical data tests were conducted such as autocorrelation function and multifractal spectrum. Also, in order to verify the accuracy of the AWMM model in representing real network traffic, simulations were carried out to analyse the byte loss rate in a single server system fed by the model-based synthetic traffic data.

Notice that we aim to establish a model that can be adaptively updated, maintaining similar modelling performance to the ‘off-line’ multiplicative cascades. The performance of those models that use all the traffic trace samples can be better in some cases, but they place a heavy burden on real-time applications.

In the simulations, we used some TCP/IP traffic traces (dec-pkt-1.tcp, dec-pkt-2.tcp, dec-pkt-3.tcp and lbl-tcp-3) collected at the Digital Equipment Corporation (DEC) and Lawrence Berkeley Laboratory (LBL) (http://ita.ee.lbl.gov/html/contrib/DEC-PKT.html) and traces collected between the years 2000 and 2002 at the Petrobras computer network [23]. We mainly considered traffic samples at an aggregation scale of 512 ms for the DEC and LBL TCP/IP traffic traces at which these traces present multifractal characteristics [23, 24]. We also used some Ethernet traffic traces downloaded from Bellcore (http://ita.ee.lbl.gov/html/contrib/BC.html) that present self-similar as well as multifractal characteristics. All the traffic traces here show the traffic volume in bytes per unit time interval.

4.1 Autocorrelation function

The autocorrelation function reflects the second-order statistics of a time series, giving an idea regarding the long-range dependence in the data process. Let \( y(t) \) be a process with mean \( \mu \), and standard deviation \( \sigma \), and its shifted version \( y(t + k) \) with mean \( \mu_{t+k} \) and standard deviation \( \sigma_{t+k} \). The correlation coefficient \( \rho(k) \) for the process \( y(t) \) is defined by

\[
\rho(k) = \frac{E[(y(t + k) - \mu_{t+k})(y(t) - \mu_t)]}{\sigma_{t+k}\sigma_t}
\]

Fig. 1 compares the correlation coefficients computed from the Bellcore Ethernet Bc-Aug traffic trace and the synthetic traces of the AWMM and MWM models. The AWMM generated series presents correlation coefficients closely matching to those computed directly from the network traffic, even for higher values of \( k \). The decay of autocorrelation function reveals the long-range-dependent characteristics of the synthetic AWMM process.

One can use the mean square error (MSE) to verify which model autocorrelation curve is closer to the network traffic one.

\[
\text{MSE} = E[(\hat{x} - x)^2]
\]

For the Bc-Aug traffic trace, we obtained MSEs equal to \( 1.8891 \times 10^{-22} \) and \( 1.2430 \times 10^{-21} \) for the autocorrelation function of the MWM and AWMM traffic processes, respectively. That is, in this experiment, a smaller error on the autocorrelation curve is achieved using the AWMM traffic synthesis procedure.

4.2 Multifractal spectra

Mathematically speaking, singularities are points (e.g. in an equation, on a curve, on a surface, etc.) at which abrupt transitions occur or the object becomes degenerated (lack of convergence). To characterise singular structures, the local regularity of a signal needs to be precisely quantified. Lipschitz exponents provide regularity measurement over time intervals, as well as for isolated signal points. If a function \( f(t) \) has a singularity at time instant \( v \), which means that this function is not differentiable at \( v \), then the Lipschitz exponent (also known as the Hölder exponent) characterises this singular behaviour [25].

**Definition 2:** A function \( Z \) is said to belong to a set \( C^a \), if there exists a polynomial \( P \) whose order is less than \( a \), and a constant \( C \) such that [25]

\[
|Z(y) - P(y)| < C|y - x|^a
\]

Then, the local Hölder exponent \( \alpha(x) \) of function \( Z \) at point \( x \) is given by

\[
\alpha(x) = \sup\{a: Z \in C^a_x\}
\]
Local Hölder exponents characterise function behaviour on a neighbourhood of a given point $x$. Multifractal processes present varying Hölder exponent values within any finite interval. The Hölder exponents describe the local scaling properties of a realised path at any time instance. Their distribution can be represented by a renormalised density function called the multifractal spectrum. In an alternative interpretation, the multifractal spectrum describes the fractal dimension of the set of time instants having a given common local exponent value [9]. The concept of Hölder exponent $\alpha$ is related to the local singularity of a process, i.e. it characterises the burstiness of a process at a given time instant. The multifractal spectrum $f(\alpha)$ of a process $X(t)$ can be obtained through the Legendre transform of $\tau(q)$ (scaling function) defined as follows

$$f(\alpha) = \min_q \{q \alpha - \tau(q)\} \quad (33)$$

Through the Legendre transform, we computed the multifractal spectrum for the AWMM and MWM processes. The multifractal spectra of the DEC traffic traces (dec-pkt-1, 2 and 3) present most of their Hölder exponent values as being less than 1. This fact characterises the high incidence of multiscale bursts. Comparatively, it can be observed that the AWMM model efficiently captures the local scaling properties of a process through its multifractal spectrum, and consequently the ‘multifractality’ of the real traffic traces (Fig. 2).

If a visual comparison for the multifractal spectra is not precise enough, additional test can be done. In this work, we make use of the Kolmogorov–Smirnov (K–S) goodness-of-fit hypothesis test. The K–S test determines if two independent random samples are drawn from the same underlying distribution [26]. In the K–S-test, the $p$-value is the probability that a sample could have been drawn from the population being tested given the assumption that the null hypothesis is true. Hence, we can apply the K–S test to the Hölder exponents of the AWMM, MWM and of real traffic processes in order to decide which model provides a higher $p$-value in the test. That is, a higher $p$-level indicates that the Hölder exponents of the model process has higher probability to share the same Hölder distribution (multifractal spectrum) of the real traffic process. For the dec-pkt-1 traffic trace, we obtained a $p$-value of 0.0177 for the AWMM, whereas a smaller value was obtained for the MWM ($p$-value equal to 0.0033).

### 4.3 Loss probability estimation

We consider a single server system with a finite size buffer fed by the AWMM process in order to compare its accuracy in representing real data traffic. We mainly analysed the byte loss rate in function of buffer utilisation. We define the buffer utilisation $\lambda$ as the ratio of the total service time to the total time for which the buffer is found in use. A higher value of $\lambda$ indicates a higher probability of data being discarded.

Let $Q$ be the queueing length process in the single server system. For a buffer size $x$, we estimate the byte loss probability $P(Q > x)$ as the ratio of the discarded number of bytes to the total number of bytes that arrived in the server [27].

Fig. 3 presents the byte loss ratio in function of the buffer utilisation for a buffer of finite size equal to 65 kB for the 10-7-S-1 Petrobrás multifractal traffic trace at the 100 ms timescale. The byte loss ratio was verified by using the average of 100 realisations of the corresponding AWMM and MWM processes. The simulation results validate the proposed AWMM approach which faithfully models the considered real Internet traffic traces by providing similar queueing behaviours.

Different from the MWM, the AWMM achieves these byte loss ratio values through an adaptive modelling. That is, the parameters of the traffic model are updated at each

---

**Figure 2** Legendre spectrum (dec-pkt-1 traffic trace)

**Figure 3** Percentual loss for a buffer size equal to 65 kB
corresponding time instant to generate the desired synthetic trace. Further, the obtained AWMM synthetic trace is used in the queueing simulation test.

5 Effective bandwidth for traffic flows

The concept of effective bandwidth for high-speed network traffic was first presented for i.i.d and on–off sources [2]. Further, methods of effective bandwidth estimation for Markov and self-similar processes were carried out [28–30].

The effective bandwidth of traffic flows can be parametrically formulated requiring the application of a suitable analytical traffic model capable of fully describing the traffic source characteristics. As an alternative, the measured effective bandwidth (MEB) is an approach of effective bandwidth computing based on the process samples. The MEB of a traffic stream is defined as [2, 31]

\[ \alpha(t, s, N_t) = \frac{1}{s} \log_2 \left( \sum_{k=1}^{N_t} \Theta_{j,k} \right) \]

where \(X(0, t)\) represents the aggregate number of packet arrivals at a time interval of length \(t\), \(\hat{E}_X[e^{\lambda X(0,t)}]\) is the measured moment generating function over an \(N_t\)-sample traffic trace and \(s\) is the space parameter that is related to the overflow probability \(P(Q > x)\) [32]. In the next section, we introduce an effective bandwidth estimation approach for multifractal processes.

5.1 Effective bandwidth for the AWMM traffic process

In this work, we explore some statistical approaches to develop an effective bandwidth expression for the AWMM-based traffic processes as summarised in the following proposition.

**Proposition 1:** The effective bandwidth of AWMM process \(X(k)\) can be expressed in terms of its corresponding multipliers \(A_{j,k}\) as

\[ \alpha(s, k) = \frac{1}{s^2} \log_2 \left( \sum_{k=1}^{N} \left| \Theta_{j,k} \right| \right) \]

where

\[ \Theta_{j,k} = 2^{j/2} [c \nu_1 - c \nu_2] \]

\[ \nu_1 = 2^{j/2} U_{0,0} \prod_{i=0}^{j-1} [1 + (-1)^i A_{j,k}] \]

\[ \nu_2 = 2^{j/2} U_{0,0} \prod_{i=0}^{j-1} [1 + (-1)^i A_{j,k+1}] \]

and \(N\) is the number of AWMM cascade stages.

**Proof:** See appendix 10.3.

Fig. 4 shows the effective bandwidths computed through (35) as well as those given by the MEB [31] and Norros effective bandwidth (NEB) [33] approaches versus the buffer size, under the target byte loss probability set to \(10^{-6}\). The traffic trace dec-pkt-2 was used here on purpose because it is monofractal with \(H = 0.8\), and as consequence, its effective bandwidth can be appropriately determined via the NEB approach, which corresponds to a monofractal based effective bandwidth formulation [33]. The MEB and our effective bandwidth method show similar results for all considered traffic traces. For small buffer size, the NEB granted a high effective bandwidth estimate value; whereas our AWMM and the MEB estimates were less sensitive to the buffer size variation.

We also simulated a link serving the multifractal 10-7-S-1 Petrobras traffic trace at timescale of 100 ms with the server capacity equal to the effective bandwidth given by (35) and a target QoS loss ratio of \(10^{-4}\). Fig. 5 shows how the packet loss ratio varies with the space parameter \(s\), maintaining the required loss ratio of \(10^{-4}\). Evidently, it can be seen that the effective bandwidths given by the proposed approach attain the loss probability requirement.

6 Adaptive effective bandwidth estimation

The AWMM has an analytical structure suitable for online traffic modelling. Taking advantage of this fact, in this section, we design a procedure for adaptive effective bandwidth computing. For online effective bandwidth estimation we need to periodically update the triple \((\alpha, \gamma, \rho)\) parameters of the AWMM obtained through the scaling function \(s(q)\) and the moment factor \(c(q)\). In addition, the value of the moment generating function \(M(k)\) of the traffic
flow must be updated at time instant \( k \). Assuming the traffic process obeys the AWMM and based on the effective bandwidth estimation formula, we can update the moment generating function of a traffic process by

\[
M(k) = \left( 1 - \frac{1}{k} \right) M(k-1) + \frac{1}{k} \sum_{i=1}^{k} 2^{-i/2} \left[ e^{v_1} - e^{v_2} \right] \tag{39}
\]

where \( v_1 \) and \( v_2 \) are given by (37) and (38), respectively.

Then, the AWMM traffic synthesis algorithm is required for the adaptive effective bandwidth estimation resulting in the following recursive algorithm.

**Proposed Algorithm 2:** On-line effective bandwidth estimation algorithm

1. Generate the cascade multipliers \( A_{\alpha,j} \) and compute the parameters \( v_1 \) and \( v_2 \) at scale \( j \) using the Proposed Algorithm 1 (Section 3).

2. Update the moment generating function using (39) at the time instant \( k = 2^j \).

3. The value of the effective bandwidth at the time instant \( k \) is given by

\[
\alpha(s, t) = \frac{1}{s} \log(M(k)) \tag{40}
\]

As a validation of the proposed online effective bandwidth estimation algorithm, we arbitrarily set the following conditions for the test: the buffer capacity \( B = 50 \text{ kB} \) and byte loss probability \( P(Q > B) \) equal to \( 10^{-7} \). Fig. (6) compares the effective bandwidth estimates for a TCP–IP traffic trace (dec-pkt-2 with Hurst parameter 0.80) given by the proposed static effective bandwidth approach and its adaptive version. The adaptive effective bandwidth closely matches that obtained via the static computing approach where the latter is based on the entire data trace. It can be noticed that the adaptive effective bandwidth tracks the static one at the considered time instant, approximately reaching the same value as it was calculated by the static scheme that uses the whole traffic data until this time instant. However, the adaptive method is a less time consuming approach, requiring minimal memory storage. In the following section, we apply the proposed online effective bandwidth computing principle to build an adaptive bandwidth provisioning scheme.

### 6.1 Adaptive bandwidth provisioning scheme

It is a challenging task to provide guaranteed QoS for network applications while simultaneously maintaining high network utilisation. In this regard, bandwidth provisioning in high-speed networks needs to be dynamic, adaptive and measurement-based to attain more efficient use of network resources. The online effective bandwidth algorithm described in the previous section was used to implement a fully adaptive bandwidth provisioning method. In our bandwidth provisioning scheme, the traffic is measured during a time slot \( t_{\text{slot}} \). Bandwidth allocation is performed at a larger timescale, every \( N_t \times t_{\text{slot}} \), known as the resizing window. We apply the adaptive effective bandwidth estimation algorithm to update the effective bandwidth of the traffic trace at every time instant of the resizing window.

In the literature, few works have dealt with adaptive bandwidth allocation with QoS guarantees [34, 35]. For comparison purposes, we also implemented the Duffield’s provision scheme, setting the needed bandwidth in resizing window \( j + 1 \) to \( c_{j+1} = m_j + d \sqrt{v_j} \), where \( m_j \) and \( v_j \) are, respectively, the mean and variance of the measured traffic load in the current window \( j \). The parameter \( d \) is related to the rate overload probability [35]. The Duffield’s provision
method assumes a model with known mean and variance values in each resizing window.

In order to quantitatively evaluate the considered provisioning schemes, we verified their link utilisation and byte loss ratio. Experimentally, we tested the proposed adaptive bandwidth provisioning scheme with various traffic traces. Tables 1–3 show the link utilisation and the byte loss ratio obtained through Duffield’s method, the measurement-based approach (static effective bandwidth provisioning) and the proposed adaptive bandwidth scheme for three different traffic traces. In the simulations, we observed that the proposed scheme generally presents a larger link utilisation than the static bandwidth provisioning, attaining the required byte loss ratio. Therefore besides being appropriate for real-time applications, the proposed scheme also achieves a higher link utilisation in comparison to the static one. Regarding Duffield’s method, one can see that it consists of a simple but efficient method. However, it can be noticed that Duffield’s method in some cases did not achieve the target byte loss probability mainly for smaller values of the buffer size and it requires in advance the knowledge of the mean and variance in each resizing window.

Further investigation is still needed in terms of the effect of the length of the time slot and resizing windows which is left for future work.

### 7 Conclusion

In this article, we first discussed the importance and applicability of the effective bandwidth and reviewed some important multifractal traffic model concepts. Then, we proposed a novel adaptive wavelet-based model based on the wavelet coefficients of multifractal processes for network traffic characterisation. Statistical and performance tests showed that the AWMM matched closely to the MWM (i.e. with non-adaptive parameter updating) as well as to real network traffic in simulations. Besides being adaptive and able to work in real time, the AWMM is more efficient than the MWM in requiring fewer input parameters. The autocorrelation function and the multifractal spectra of the AWMM-based synthetic traces revealed their long-range dependence and multifractal characteristics as encountered in the real network traffic traces. The queueing test results complemented our analysis of the generated synthetic traces in order to show that the AWMM is adequate for real traffic modelling and network performance evaluation.

Using a convergence theory and other related statistical approaches, we derived an effective bandwidth function for the AWMM, extendable to other cascade processes. The experimental results showed that the AWMM-based effective bandwidth estimate is much more realistic than the Norros’ monofractal one, not only guaranteeing the required loss probability for multifractal traffic processes, but also achieving higher link utilisation.

We also developed an adaptive effective bandwidth computing algorithm. Based on it, we incorporated the online updating virtues of the AWMM into an adaptive bandwidth provisioning scheme. Experimental results validated the proposed provisioning scheme showing its advantage of achieving higher link utilisation than the static one and also guaranteeing the required loss probability. Therefore both the adaptive wavelet-based modelling and the bandwidth provisioning scheme proposed in this work are powerful tools that can be promptly applied to network traffic control and design, contributing towards really providing QoS to network traffic flows.

### 8 Acknowledgment

The authors would like to thank Fapesp (Proc. 06/60363-6) for the financial support.

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<tr>
<th>Table 1 Byte loss probability and link utilisation for the dec-pkt-1 traffic trace</th>
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<td>Scheme</td>
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<td>static</td>
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<td>Duffield</td>
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<td>proposed</td>
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\( N_t = 100, t_{\text{slot}} = 10 \text{ ms}, B = 5 \text{ kB} \) and target loss probability equals to \( 10^{-3} \)

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<th>Table 2 Byte loss probability and link utilisation for the 10-7-5-1 Petrobrás traffic trace</th>
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<tr>
<td>Duffield</td>
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<td>proposed</td>
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\( N_t = 100, t_{\text{slot}} = 100 \text{ ms}, B = 10 \text{ kB} \) and target loss probability equals to \( 10^{-4} \)

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<th>Table 3 Byte loss probability and link utilisation for the 3-7-1-1 Petrobrás traffic trace</th>
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<td>Scheme</td>
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<td>Duffield</td>
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<td>proposed</td>
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\( N_t = 1000, t_{\text{slot}} = 10 \text{ ms}, B = 50 \text{ kB} \) and target loss probability equals to \( 10^{-3} \)
9 References


of $X(\delta)$ becomes

$$\text{var}[X^m] = E[2^{-j} \sum_{k=(j-1)m+1}^{jm} Y(\delta)\mu(k)]^2 - E^2[2^{-j} \sum_{k=(j-1)m+1}^{jm} Y(\delta)\mu(k)]$$

Thus, we can write the second moment of the wavelet coefficients obtained from the discrete wavelet transform in the Haar wavelet domain of a traffic process can be determined as

$$E(W_j^2) = \frac{\sum_{i=0}^{2^j-1} \langle X(i), X(i+j) \rangle}{2^j} - 2 \sum_{i=0}^{2^j-1} \langle X(i), X(i+j) \rangle^{2j}$$

By using the knowledge of the moments of the cascade multipliers $A$, the moments of the lognormal random variable $Y$ and (43), the variance of the aggregated process

$$\text{var}[X^m] = 2^{-2j}E[Y^2]E[A^2]\left(\frac{\alpha+1}{\alpha+1/2}\right)^2 - 2^{-2j}E^2[Y]\left(\frac{\alpha+1}{\alpha+1/2}\right)^2$$

$$\text{var}[X^m] = 2^{-2j}\left(\frac{\alpha+1}{\alpha+1/2}\right)^2$$

### 10.2 Proof of Lemma 2

Let $X^m$ be the $m$-order aggregated process of $X(\delta)$. For $m = 2^j$, the variance of the aggregated process $X^{(2j)}$ can be written as

$$\alpha_{X^{(2j)}}^2 = \frac{\sum_{i=0}^{N_t-2} \sum_{j=0}^{N_t-1} \langle X(i), X(i+1/2) - X \rangle^2}{N_t/2}$$

where $X$ and $N_t$ are the mean and the number of samples of the traffic process. Now, let $m = 2^j$, the variance of the $2^j$-order aggregated process is

$$\alpha_{X^{(2j)}}^2 = \frac{1}{N_t} \sum_{i=0}^{N_t-1} \langle X(i), X(i+j) \rangle^2 + \frac{1}{2j-1} Z_j - \overline{X}^2$$

where $Z_j = \sum_{i=0}^{N_t-1} \langle X(i), X(i+j) \rangle / N_t$. Using the fact that

$$\frac{1}{2^j} \sum_{i=0}^{2^j-1} \langle X(i), X(i+j) \rangle^2 = e^{2\rho+2\gamma} \left(\frac{\alpha+1}{\alpha+1/2}\right)^2$$

$Z_j$ can be alternatively expressed as

$$Z_j = 2^{j-1} \left[ \alpha_{X^{(2j)}}^2 - e^{2\rho+2\gamma} \left(\frac{\alpha+1}{\alpha+1/2}\right)^2 \right]$$

According to [36], the second moment of wavelet coefficients obtained from the discrete wavelet transform in the Haar wavelet domain of a traffic process can be determined as

$$E(W_j^2) = \frac{\sum_{i=0}^{2^j-1} \langle X(i), X(i+j) \rangle}{2^j} - 2 \sum_{i=0}^{2^j-1} \langle X(i), X(i+j) \rangle^{2j}$$

Thus, we can write the second moment of the wavelet coefficients $E(W_j^2)$ as

$$E(W_j^2) = e^{2\rho+2\gamma} \left(\frac{\alpha+1}{\alpha+1/2}\right)^2 - 2Z_j$$

where $Z_j$ is given by (48).
The scaling coefficients $U_{N,k}$ at the coarsest scale $j = N$ are computed as the following:

$$U_{N,k} = 2^{-N/2} \sum_{i=2^N/k}^{2^{N+k-1}-1} X(i) \quad k = 0, 1, \ldots, N - 1$$ (51)

Then, the mean of the scaling coefficients at the coarsest scale $j = N$ of the AWMM process is given by

$$E[U_{N,k}] = 2^{-N/2} \frac{\sum_{i=0}^{N-1} X(i)}{N} = 2^{N/2} E[X(i)]$$

$$E[U_{N,k}] = 2^{-N/2} (e^{\alpha+\gamma}/2)$$ (52)

Similarly, the variance of the scaling coefficients at the coarsest scale $j = N$ of the AWMM process can be expressed as

$$\sigma^2_{U_{N,k}} = \frac{\sum_{i=0}^{N-1} X(i)^2}{N} + 2Z_N = (E[U_{N,k}])^2$$

$$\sigma^2_{U_{N,k}} = e^{2\alpha+2\gamma} \left( \frac{\alpha + 1}{\alpha + 1/2} \right)^2 - 2^9/2Z_N - 2^{-N} e^{2\alpha+2\gamma}$$ (53)

Therefore inserting $Z_N$ (48) into (53), we can calculate the values of the variance of the wavelet coefficients.

### 10.3 Proof of Proposition 1

First, let us turn to a more precise mathematical specification of multiplicative cascades. Consider repeated splittings of $T$, a unit cube $R^d$ for some $d \geq 1$, into a number $b^N$ ($b > 2$), sub-cubes (pixels) of volume $b^{-n}$, $n = 1, 2, \ldots$, such that the $n$th-stage mass of the pixel $D(t_1, t_2, \ldots, t_n)$ is given by the random measure $\mu_n(D(t_1, t_2, \ldots, t_n)) = b^{-\sum_{k=1}^{n} k t_k}$, where $A_{i:j}$ is the cascade generators, an i.i.d measure of non-negative, unit mean random variables indexed by the pixel addresses $v$ at different fine scales $b^{-n}$ for $n \geq 1$. The cascade measure $\mu_{\omega}$ is obtained as the limit of the sequence $\mu_n$ as $n \to \infty$.

The structure function or the modified cumulant generating function for log $A_{i:j}$ is defined as [37]

$$\Omega_{i,j}(b) = \log b E[A_{i:j} \log(1 + A_{i:j})] - (b - 1)$$ (54)

which parameterises the distribution of $A_{i:j}$. It has been shown that the function $\Omega_{i,j}(b)$ can be estimated by [37]

$$\xi_{i,j}^b = \frac{1}{n} \log b \sum_{A_{i:j}} \mu_{\omega}(A_{i:j})$$ (55)

That is, $\xi_{i,j}^b$ converges to $\Omega_{i,j}(b)$ as $n \to \infty$ for all $b > 0$ and cascades that $E[\mu_{\omega}(T)] > 0$.

Alternatively, the structure function of a cascade process with $j$ stages, denoted as $\xi(q,j)$, can also be obtained by the following equations [16, 36]

$$\xi(q,j) = \sum_{k=1}^{2^j} [W_{j,k}]^g$$ (56)

$$\xi(q,j) = \frac{\log (2\xi(q,j))}{j}$$ (57)

where $W_{j,k}$ represents the wavelet coefficients of the traffic process.

Let the process $g$ be the exponential version of a multifractal cascade multiplied by $s$. Then, we can express this new process as

$$g_m(\Delta, t_1, t_2, \ldots, t_n) = e^{\mu_m(\Delta, t_1, t_2, \ldots, t_n)}$$ (58)

In case of Haar wavelets, the wavelet coefficients $W_{j,k}$ of a stochastic process $X$ are explicitly given by

$$W_{j,k} = 2^{-j/2}(X(2j) - X(2j + 1))$$ (59)

For the AWMM, expressions (14) and (16) hold. Then, the cascade coefficients $\Theta_{j,k}$ for the process $g$ given by (58) can be written as

$$\Theta_{j,k} = 2^{-j/2}[e^{\alpha_t} - e^{\alpha_t}]$$ (60)

where

$$v_1 = 2^{-j/2} U_{0,0} \prod_{i=0}^{j-1} [1 + (-1)^i A_{2i},]$$ (61)

and

$$v_2 = 2^{-j/2} U_{0,0} \prod_{i=0}^{j-1} [1 + (-1)^i A_{2i+1}]$$ (62)

According to the large deviation theory, the effective bandwidth can be expressed as [2]:

$$\alpha(s, t) = -\frac{1}{\mu} \log E[e^{\lambda(X(t+\tau)\alpha - \lambda_t)]} = -\frac{1}{\mu} \log \frac{1}{p} \sum_{k=1}^{p} e^{\lambda(X(t+\tau)\alpha)$$

or

$$\alpha(s, t) = -\frac{1}{\mu} \left( \log \frac{1}{p} + \log \sum_{k=1}^{p} e^{\lambda(X(t+\tau)\alpha)\right}$$ (63)

where $\tau \geq 0, p = 2N$ is the number of samples of $X$ and $N$ is the number of cascade stages. Notice that the term $\log \sum_{k=1}^{p} e^{\lambda(X(t+\tau)\alpha)$ of (63) can be estimated using (55) applied to the process $g$. Once having the structure function
estimated via \( \hat{Z}(q,j) \), the second part of the right side of \( \alpha(s,t) \) can be estimated by \( 2^N \hat{\xi}(1,j) \) and consequently

\[
\hat{\xi}(1,j) = \log \left( \sum_{k=1}^{2^l} |W_{j,k}|^2 \right)
\]

(64)

Finally, inserting (60)–(62) into (64) we can compute the effective bandwidth \( \alpha(s,t) \) using the multipliers \( A_{j,k} \) by the following equation

\[
\alpha(s,t) = \frac{1}{st} \left( \log \frac{\sum_{k=1}^{2^N} |\Theta_{j,k}|^2}{2^N} \right)
\]

(65)

where \( t \) is dyadic (\( t = 2^k \)) and \( k = 1, \ldots, 2^N \).