IE550 – Processmamento Digital de Sinais – 2ª Lista de Exercícios

Obs.: Exercícios selecionados do Oppenheim e Schafer, *Discrete-Time Signal Processing.*

5.25. Consider a causal linear time-invariant system with system function

$$H(z) = \frac{1 - a^{-1}z^{-1}}{1 - az^{-1}},$$

where a is real.

(a) Write the difference equation that relates the input and the output of this system.

(b) For what range of values of a is this system stable?

(c) For $a = \frac{1}{2}$, plot the pole-zero diagram and shade the region of convergence.

(d) Find the impulse response $h[n]$ for this system.

(e) Show that this system is an allpass system, i.e., that the magnitude of the frequency response is a constant. Also, specify the value of the constant.

5.32. Consider the linear time-invariant system whose system function is

$$H(z) = (1 - 0.9e^{j0.6\pi}z^{-1})(1 - 0.9e^{-j0.6\pi}z^{-1})(1 - 1.25e^{j0.8\pi}z^{-1})(1 - 1.25e^{-j0.8\pi}z^{-1}).$$

(a) Find all causal system functions that result in the same frequency-response magnitude as $H(z)$ and for which the impulse responses are real-valued and of the same length as the impulse response associated with $H(z)$. (There are four different such system functions.) Explicitly identify which one is minimum phase and which, to within a time shift, is maximum phase.

(b) Find the impulse responses for the system functions in part (a).

(c) For each of the sequences in part (b), compute and plot the quantity

$$E[n] = \sum_{m=0}^{n} (h[m])^2$$

for $0 \leq n \leq 5$. Indicate explicitly which plot corresponds to the minimum-phase system.

5.40. Consider the class of discrete-time filters whose frequency response has the form

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\alpha\omega},$$

where $|H(e^{j\omega})|$ is a real and nonnegative function of ω and α is a real constant. As discussed in Section 5.7.1, this class of filters is referred to as *linear phase* filters.

Also consider the class of discrete-time filters whose frequency response has the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},$$

where $A(e^{j\omega})$ is a real function of ω, α is a real constant, and β is a real constant. As discussed in Section 5.7.2, filters in this class are referred to as *generalized linear phase* filters.

For each of the filters in Fig. P5.40, determine whether it is a generalized linear phase filter. If it is, then find $A(e^{j\omega})$, α, and β and indicate if it is also a linear phase filter.

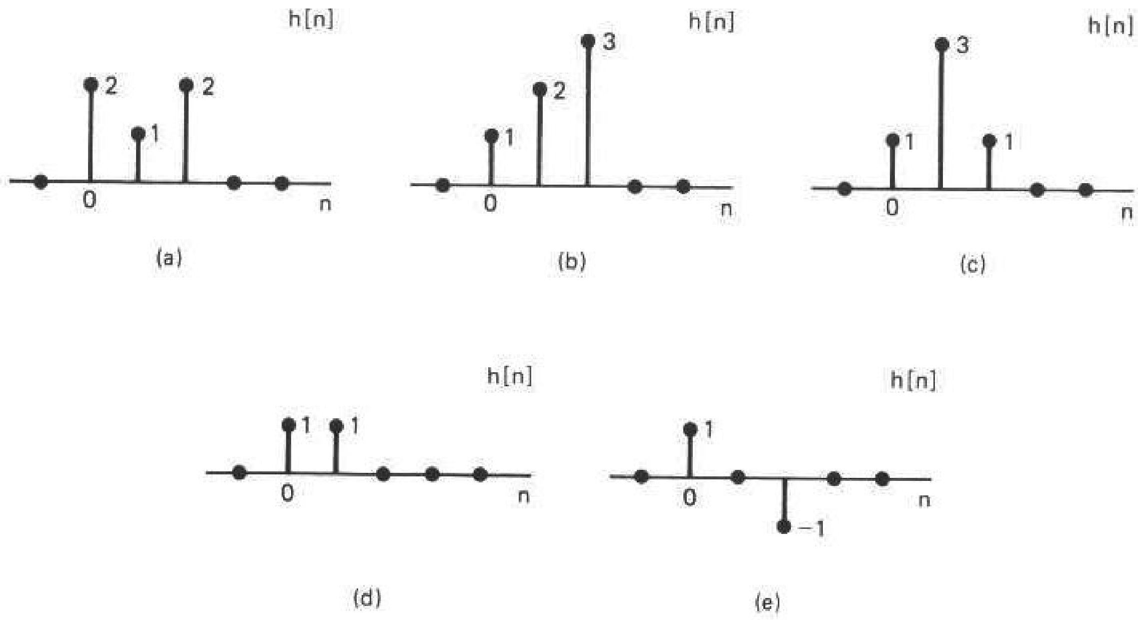

Figure P5.40

5.45. Consider the class of FIR filters that have $h[n]$ real, $h[n] = 0$ for $n < 0$ and $n > M$, and one of the following symmetry properties:

$$\text{Symmetric:} \quad h[n] = h[M - n]$$

$$\text{Antisymmetric:} \quad h[n] = -h[M - n]$$

All filters in this class have generalized linear phase, i.e., have frequency response of the form

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega + j\beta},$$

where $A(e^{j\omega})$ is a real function of ω, α is a real constant, and β is a real constant.

For the following table, show that $A(e^{j\omega})$ has the indicated form, and find the values of α and β.

Type	Symmetry	Filter Length $(M + 1)$	Form of $A(e^{j\omega})$	α	β
I	Symmetric	Odd	$\sum_{n=0}^{M/2} a[n]\cos \omega n$		
II	Symmetric	Even	$\sum_{n=1}^{(M+1)/2} b[n]\cos \omega(n - 1/2)$		
III	Antisymmetric	Odd	$\sum_{n=1}^{M/2} c[n]\sin \omega n$		
IV	Antisymmetric	Even	$\sum_{n=1}^{(M+1)/2} d[n]\sin \omega(n - 1/2)$		

Here are several helpful suggestions.

- For type I filters, first show that $H(e^{j\omega})$ can be written in the form

$$H(e^{j\omega}) = \sum_{n=0}^{(M-2)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-2)/2} h[M-n]e^{-j\omega[M-n]} + h[M/2]e^{-j\omega(M/2)}.$$

- The analysis for type III filters is very similar to that for type I, with the exception of a sign change and removal of one of the above terms.
- For type II filters, first write $H(e^{j\omega})$ in the form

$$H(e^{j\omega}) = \sum_{n=0}^{(M-1)/2} h[n]e^{-j\omega n} + \sum_{n=0}^{(M-1)/2} h[M-n]e^{-j\omega[M-n]}$$

and then pull out a common factor of $e^{-j\omega(M/2)}$ from both sums.
- The analysis for type IV filters is very similar to that for type II filters.

6.5. For the system function

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}},$$

draw the flow graphs of all possible realizations for this system as cascades of first-order systems.

6.7. Consider a causal linear time-invariant system whose system function is

$$H(z) = \frac{1 + \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}.$$

(a) Draw the signal flow graphs for implementations of the system in each of the following forms:
- **(i)** Direct form I
- **(ii)** Direct form II
- **(iii)** Cascade form using first- and second-order direct form II sections
- **(iv)** Parallel form using first- and second-order direct form II sections
- **(v)** Transposed direct form II

(b) Write the difference equations for the flow graph of part (v) in (a) and show that this system has the correct system function.

6.12. A linear time-invariant system with system function

$$H(z) = \frac{0.2(1 + z^{-1})^6}{(1 - 2z^{-1} + \frac{7}{8}z^{-2})(1 + z^{-1} + \frac{1}{2}z^{-2})(1 - \frac{1}{2}z^{-1} + z^{-2})}$$

is to be implemented using a flow graph of the form shown in Fig. P6.12.

(a) Fill in all the coefficients in the diagram of Fig. P6.12. Is your solution unique?

(b) Define appropriate node variables in Fig. P6.12 and write the set of difference equations that is represented by the flow graph.

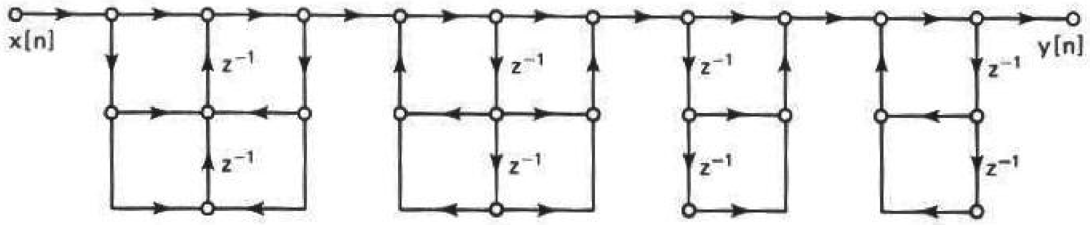

Figure P6.12

6.15. The impulse response of a linear time-invariant system is

$$h[n] = \begin{cases} a^n, & 0 \le n \le 7, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Draw the flow graph of a direct form FIR implementation of the system.

(b) Show that the corresponding system function can be expressed as

$$H(z) = \frac{1 - a^8 z^{-8}}{1 - az^{-1}}, \qquad |z| > |a|.$$

(c) Draw the flow graph of another system having the same system function and consisting of a cascade of an FIR system with an IIR system (assume $|a| < 1$).

(d) Which implementation of the system requires
  (i) the most storage (delay elements)?
  (ii) the most arithmetic (multiplications and additions per output sample)?

6.19. Figure P6.19 shows the direct form and lattice form flow graphs for the FIR system discussed in Example 6.7. We wish to verify that the two flow graphs have the same system function.

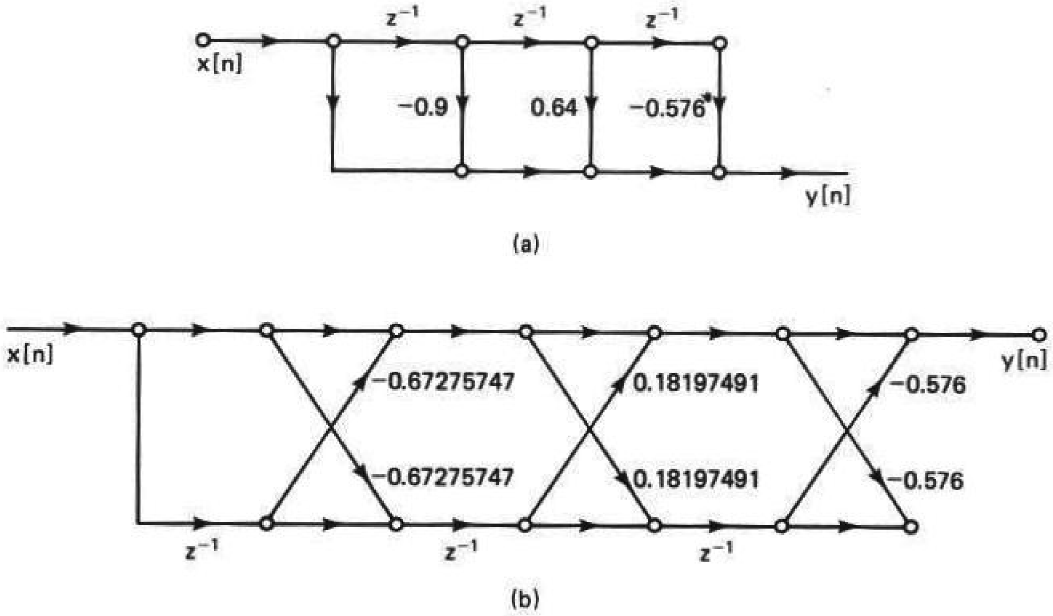

Figure P6.19

(a) Use Eqs. (6.54) to find the coefficients of the system function polynomial $A(z)$ for the system of Fig. P6.19(b) from the k-parameters and compare $A(z)$ to the system function of the system in Fig. P6.19(a).

(b) Compute the impulse responses of the two systems in Fig. P6.19 by simply tracing an impulse input through all paths in the flow graphs and summing the impulses that arrive at the output with the same delay.

Obs. Eqs (6.54) are Levinson-Durbin relations.

7.1. Consider a continuous-time system with impulse response $h_c(t)$ and system function

$$H_c(s) = \frac{s+a}{(s+a)^2 + b^2}.$$

(a) Use impulse invariance to determine $H_1(z)$ for a discrete-time system such that $h_1[n] = h_c(nT)$.

(b) Use step invariance to determine $H_2(z)$ for a discrete-time system such that $s_2[n] = s_c(nT)$, where

$$s_2[n] = \sum_{k=-\infty}^{n} h_2[k] \qquad \text{and} \qquad s_c(t) = \int_{-\infty}^{t} h_c(\tau)d\tau.$$

(c) Determine the step response $s_1[n]$ of system 1 and the impulse response $h_2[n]$ of system 2. Is it true that $h_2[n] = h_1[n] = h_c(nT)$? Is it true that $s_1[n] = s_2[n] = s_c(nT)$?

7.4. A discrete-time lowpass filter is to be designed by applying the impulse invariance method to a continuous-time Butterworth filter having magnitude-squared function

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}.$$

The specifications for the discrete-time system are those of Example 7.3, i.e.,

$$0.89125 \le |H(e^{j\omega})| \le 1, \qquad 0 \le |\omega| \le 0.2\pi,$$

$$|H(e^{j\omega})| \le 0.17783, \qquad 0.3\pi \le |\omega| \le \pi.$$

Assume as in Example 7.3 that aliasing will not be a problem; i.e., design the continuous-time Butterworth filter to meet passband and stopband specifications as determined by the desired discrete-time filter.

(a) Sketch the tolerance bounds on the magnitude of the frequency response $|H_c(j\Omega)|$ of the continuous-time Butterworth filter such that after applying the impulse invariance method (i.e., $h[n] = T_d h_c(nT_d)$), the resulting discrete-time filter will satisfy the given design specifications. Do not assume that $T_d = 1$ as in Example 7.3.

(b) Determine the integer order N and the quantity $T_d\Omega_c$ such that the continuous-time Butterworth filter exactly meets the specifications determined in part (a) at the passband edge.

(c) Note that if $T_d = 1$, your answer in part (b) should give the values of N and Ω_c obtained in Example 7.3. Use this observation to determine the system function $H_c(s)$ for $T_d \ne 1$ and to argue that the system function $H(z)$ that results from impulse invariance design with $T_d \ne 1$ is the same as the result for $T_d = 1$ given by Eq. (7.19).

7.8. The system function of a discrete-time system is

$$H(z) = \frac{2}{1 - e^{-0.2}z^{-1}} - \frac{1}{1 - e^{-0.4}z^{-1}}.$$

(a) Assume that this discrete-time filter was designed by the impulse invariance method with $T_d = 2$; i.e., $h[n] = 2h_c(2n)$, where $h_c(t)$ is real. Find the system function $H_c(s)$ of a continuous-time filter that could have been the basis for the design. Is your answer unique? If not, find another system function $H_c(s)$.

(b) Assume that $H(z)$ was obtained by the bilinear transform method with $T_d = 2$. Find the system function $H_c(s)$ that could have been the basis for the design. Is your answer unique? If not, find another $H_c(s)$.

7.17. Consider a continuous-time lowpass filter $H_c(s)$ with passband and stopband specifications

$$1 - \delta_1 \le |H_c(j\Omega)| \le 1 + \delta_1, \qquad |\Omega| \le \Omega_p,$$
$$|H_c(j\Omega)| \le \delta_2, \qquad \Omega_s \le |\Omega| \le \pi.$$

This filter is transformed to a lowpass discrete-time filter $H_1(z)$ by the transformation

$$H_1(z) = H_c(s)\Big|_{s=(1-z^{-1})/(1+z^{-1})},$$

and the same continuous-time filter is transformed to a highpass discrete-time filter by the transformation

$$H_2(z) = H_c(s)\Big|_{s=(1+z^{-1})/(1-z^{-1})}.$$

(a) Determine a relationship between the passband cutoff frequency Ω_p of the continuous-time lowpass filter and the passband cutoff frequency ω_{p1} of the discrete-time lowpass filter.

(b) Determine a relationship between the passband cutoff frequency Ω_p of the continuous-time lowpass filter and the passband cutoff frequency ω_{p2} of the discrete-time highpass filter.

(c) Determine a relationship beween the passband cutoff frequency ω_{p1} of the discrete-time lowpass filter and the passband cutoff frequency ω_{p2} of the discrete-time highpass filter.

(d) The network in Fig. P7.17 depicts an implementation of the discrete-time lowpass filter with system function $H_1(z)$. The coefficients A, B, C, and D are real. How should these coefficients be modified to obtain a network that implements the discrete-time highpass filter with system function $H_2(z)$?

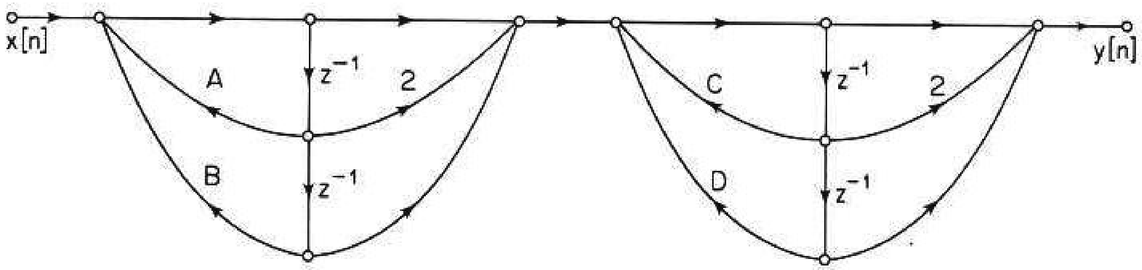

Figure P7.17

7.20. A discrete-time highpass filter can be obtained from a continuous-time lowpass filter by the following transformation:

$$H(z) = H_c(s)\Big|_{s=[(1+z^{-1})/(1-z^{-1})]},$$

(a) Show that the above transformation maps the $j\Omega$-axis of the s-plane onto the unit circle of the z-plane.

(b) Show that if $H_c(s)$ is a rational function with all its poles inside the left-half s-plane, then $H(z)$ will be a rational function with all its poles inside the unit circle of the z-plane.

(c) Suppose a desired highpass discrete-time filter has specifications

$$|H(e^{j\omega})| \le 0.01, \qquad |\omega| \le \pi/3,$$
$$0.95 \le |H(e^{j\omega})| \le 1.05, \qquad \pi/2 \le |\omega| \le \pi.$$

Determine the specifications on the continuous-time lowpass filter so that the desired highpass discrete-time filter results from the above transformation.

7.27. Consider the following ideal frequency response for a multiband filter:

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega M/2}, & 0 \le |\omega| < 0.3\pi, \\ 0, & 0.3\pi < |\omega| < 0.6\pi, \\ 0.5e^{-j\omega M/2}, & 0.6\pi < |\omega| \le \pi. \end{cases}$$

The impulse response $h_d[n]$ is multiplied by a Kaiser window with $M = 48$ and $\beta = 3.68$, resulting in a linear phase FIR system with impulse response $h[n]$.

(a) What is the delay of the filter?

(b) Determine the ideal impulse response $h_d[n]$.

(c) Determine the set of approximation error specifications that is satisfied by the FIR filter; i.e., determine the parameters δ_1, δ_2, δ_3, B, C, ω_{p1}, ω_{s1}, ω_{s2}, and ω_{p2} in

$$B - \delta_1 \le |H(e^{j\omega})| \le B + \delta_1, \qquad 0 \le \omega \le \omega_{p1},$$
$$|H(e^{j\omega})| \le \delta_2, \qquad \omega_{s1} \le \omega \le \omega_{s2},$$
$$C - \delta_3 \le |H(e^{j\omega})| \le C + \delta_3, \qquad \omega_{p2} \le \omega \le \pi.$$

8.4. In Section 8.2.2, we stated the property that

$$\text{if} \quad \tilde{x}_1[n] = \tilde{x}[n-m],$$
$$\text{then} \quad \tilde{X}_1[k] = W_N^{km}\tilde{X}[k],$$

where $\tilde{X}[k]$ and $\tilde{X}_1[k]$ are the DFS coefficients of $\tilde{x}[n]$ and $\tilde{x}_1[n]$, respectively. In this problem, we consider the proof of this property.

(a) Using Eq. (8.11) together with an appropriate substitution of variables, show that $\tilde{X}_1[k]$ can be expressed as

$$\tilde{X}_1[k] = W_N^{km} \sum_{r=-m}^{N-1-m} \tilde{x}[r]W_N^{kr}. \qquad \text{(P8.4-1)}$$

(b) The summation in Eq. (P8.4-1) can be rewritten as

$$\sum_{r=-m}^{N-1-m} \tilde{x}[r]W_N^{kr} = \sum_{r=-m}^{-1} \tilde{x}[r]W_N^{kr} + \sum_{r=0}^{N-1-m} \tilde{x}[r]W_N^{kr}. \qquad \text{(P8.4-2)}$$

Using the fact that $\tilde{x}[r]$ and W_N^{kr} are both periodic, show that

$$\sum_{r=-m}^{-1} \tilde{x}[r]W_N^{kr} = \sum_{N-m}^{N-1} \tilde{x}[r]W_N^{kr}. \qquad \text{(P8.4-3)}$$

(c) From your results in parts (a) and (b), show that

$$\tilde{X}_1[k] = W_N^{km} \sum_{r=0}^{N-1} \tilde{x}[r]W_N^{kr} = W_N^{km}\tilde{X}[k].$$

8.11. Compute the DFT of each of the following finite-length sequences considered to be of length N (where N is even).

(a) $x[n] = \delta[n]$

(b) $x[n] = \delta[n - n_0], \quad 0 \le n_0 \le N-1$

(c) $x[n] = \begin{cases} 1, & n \text{ even}, \quad 0 \le n \le N-1 \\ 0, & n \text{ odd}, \quad 0 \le n \le N-1 \end{cases}$

(d) $x[n] = \begin{cases} 1, & 0 \le n \le N/2 - 1 \\ 0, & N/2 \le n \le N-1 \end{cases}$

(e) $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$

8.24. Consider the finite-length sequence $x[n]$ in Fig. P8.24. The 4-point DFT of $x[n]$ is denoted $X[k]$. Plot the sequence $y[n]$ whose DFT is

$$Y[k] = W_4^{3k}X[k].$$

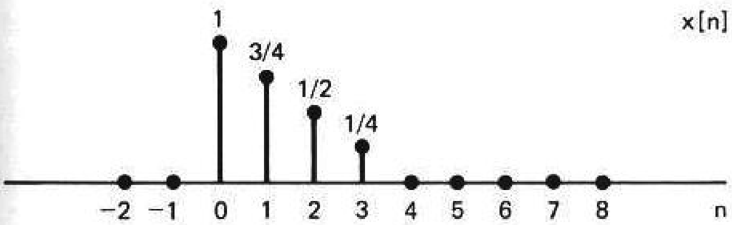

Figure P8.24

8.32. Suppose we have two 4-point sequences $x[n]$ and $h[n]$ as follows:

$$x[n] = \cos\left(\frac{\pi n}{2}\right), \qquad n = 0, 1, 2, 3,$$

$$h[n] = 2^n, \qquad n = 0, 1, 2, 3.$$

(a) Calculate the 4-point DFT $X[k]$.
(b) Calculate the 4-point DFT $H[k]$.
(c) Calculate $y[n] = x[n]④h[n]$ by doing the circular convolution directly.
(d) Calculate $y[n]$ of part (c) by multiplying the DFTs of $x[n]$ and $h[n]$ and performing an inverse DFT.

8.48. We want to implement the linear convolution of a 10,000-point sequence with an FIR impulse response that is 100 points long. The convolution is to be implemented by using DFTs and inverse DFTs of length 256.

(a) If the overlap-add method is used, what is the minimum number of 256-point DFTs and the minimum number of 256-point inverse DFTs needed to implement the convolution for the entire 10,000-point sequence? Justify your answer.

(b) If the overlap-save method is used, what is the minimum number of 256-point DFTs and the minimum number of 256-point inverse DFTs needed to implement the convolution for the entire 10,000-point sequence? Justify your answer.

(c) We will see in Chapter 9 that when N is a power of 2, an N-point DFT or inverse DFT requires $(N/2)\log_2 N$ complex multiplications and $N \log_2 N$ complex additions. For the same filter and impulse response length considered in parts (a) and (b), compare the number of arithmetic operations (multiplications and additions) required in the overlap-add method, the overlap-save method, and direct convolution.

9.9. A modified FFT algorithm called the *split-radix FFT* or SRFFT was proposed by Duhamel and Hollman (1984) and Duhamel (1986). The flow graph for the split-radix algorithm is similar to the radix-2 flow graph, but it requires fewer real multiplications. In this problem we illustrate the principles of the SRFFT for computing the DFT $X[k]$ of a sequence $x[n]$ of length N.

(a) Show that the even-indexed terms of $X[k]$ can be expressed as the $N/2$-point DFT

$$X[2k] = \sum_{n=0}^{(N/2)-1} (x[n] + x[n + N/2])W_N^{2kn}$$

for $k = 0, 1, \ldots, (N/2) - 1$.

(b) Show that the odd-indexed terms of the DFT $X[k]$ can be expressed as the $N/4$-point DFTs

$$X[4k+1] = \sum_{n=0}^{(N/4)-1} \{(x[n] - x[n + N/2]) - j(x[n + N/4] - x[n + 3N/4])\} W_N^n W_N^{4kn}$$

for $k = 0, 1, \ldots, (N/4) - 1$, and

$$X[4k+3] = \sum_{n=0}^{(N/4)-1} \{(x[n] - x[n + N/2]) - j(x[n + N/4] - x[n + 3N/4])\} W_N^{3n} W_N^{4kn}$$

for $k = 0, 1, \ldots, (N/4) - 1$.

(c) The flow graph in Fig. P9.9 represents this decomposition of the DFT for a 16-point transform. Redraw this flow graph labeling each branch with the appropriate multiplier coefficient.

(d) Determine the number of real multiplications required to implement the 16-point transform when the SRFFT principle is applied to compute the other DFTs in Fig. P9.9. Compare this number to the number of real multiplications required to

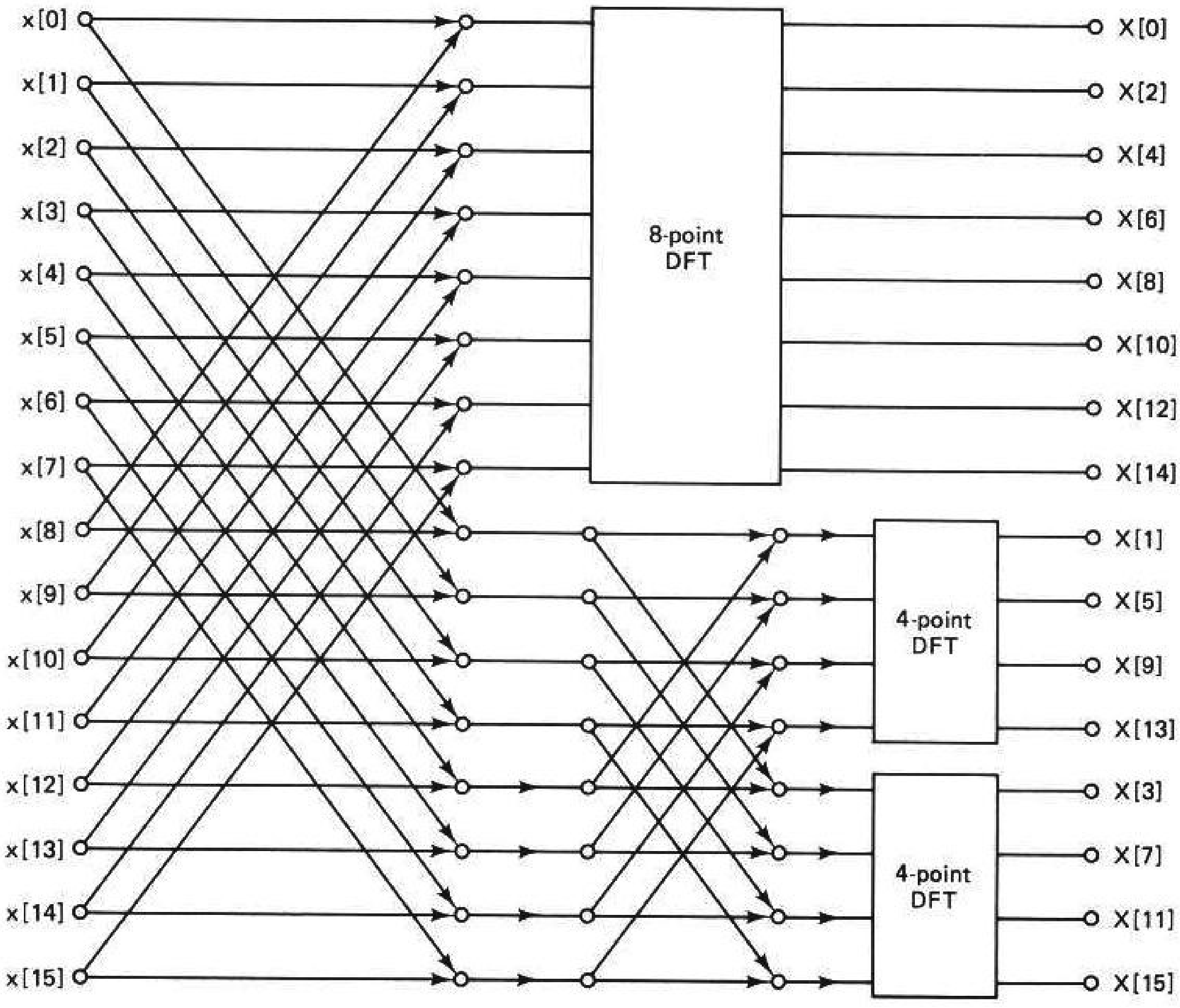

Figure P9.9

implement a 16-point radix-2 decimation-in-frequency algorithm. In both cases assume that multiplications by W_N^0 are not done.

9.14. In computing the DFT it is necessary to multiply a complex number by another complex number whose magnitude is unity, i.e., $(X + jY)e^{j\theta}$. Clearly, such a complex multiplication changes only the angle of the complex number, leaving the magnitude unchanged. For this reason, multiplications by a complex number $e^{j\theta}$ are sometimes called *rotations*. In DFT or FFT algorithms many different angles θ may be needed. However, it may be undesirable to store a table of all required values of $\sin\theta$ and $\cos\theta$, and computing these functions by a power series requires many multiplications and additions. With the CORDIC algorithm given by Volder (1959), the product $(X + jY)e^{j\theta}$ can be evaluated efficiently by a combination of additions, binary shifts, and table look-up from a small table.

(a) Define $\theta_i = \arctan(2^{-i})$. Show that any angle $0 < \theta < \pi/2$ can be represented as

$$\theta = \sum_{i=0}^{M-1} \alpha_i \theta_i + \epsilon = \hat{\theta} + \epsilon,$$

where $\alpha_i = \pm 1$ and the error ϵ is bounded by

$$|\epsilon| \leq \arctan(2^{-M}).$$

(b) The angles θ_i may be computed in advance and stored in a small table of length M. State an algorithm for obtaining the sequence $\{\alpha_i\}$ for $i = 0, 1, \ldots, M-1$ such that $\alpha_i = \pm 1$. Use your algorithm to determine the sequence $\{\alpha_i\}$ for representing the angle $\theta = 100\pi/512$ when $M = 11$.

(c) Using the result of part (a), show that the recursion

$$\begin{aligned} X_0 &= X, \\ Y_0 &= Y, \\ X_i &= X_{i-1} - \alpha_{i-1} Y_{i-1} 2^{-i+1}, \qquad i = 1, 2, \ldots, M, \\ Y_i &= Y_{i-1} + \alpha_{i-1} X_{i-1} 2^{-i+1}, \qquad i = 1, 2, \ldots, M, \end{aligned}$$

will produce the complex number

$$(X_M + jY_M) = (X + jY) G_M e^{j\hat{\theta}},$$

where $\hat{\theta} = \sum_{i=0}^{M-1} \alpha_i \theta_i$ and G_M is real, positive, and does not depend on θ. That is, the original complex number is rotated in the complex plane by an angle $\hat{\theta}$ and magnified by the constant G_M.

(d) Determine the magnification constant G_M as a function of M.

9.17. The decimation-in-time FFT algorithm was developed in Section 9.3 for radix 2, i.e., $N = 2^v$. A similar approach leads to a radix-3 algorithm when $N = 3^v$.

(a) Draw a flow graph for a 9-point decimation-in-time FFT algorithm using a 3×3 decomposition of the DFT.

(b) For $N = 3^v$, how many complex multiplications by powers of W_N are needed to compute the DFT of an N-point complex sequence using a radix-3 decimation-in-time FFT algorithm?

(c) For $N = 3^v$, is it possible to use in-place computation for the radix-3 decimation-in-time algorithm?

9.35. The input and output of a linear time-invariant system satisfy a difference equation of the form

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k].$$

Assume that an FFT program is available for computing the DFT of any finite-length sequence of length $N = 2^\nu$. Describe a procedure that utilizes the available FFT program to compute

$$H(e^{j(2\pi/512)k}) \qquad \text{for } k = 0, 1, \ldots, 512,$$

where $H(z)$ is the system function of the system.