REVISÃO

A

PROBABILIDADE

PROCESSOS ESTOCÁSTICOS

CONJUNTO: COLEÇÃO DE ELEMENTOS

SUBCONJUNTO: $B \in A$

ESPAÇO S: UNIÃO DE TODOS OS CONJUNTOS

$$A = \{a_1, a_2, \dots, a_m\}$$

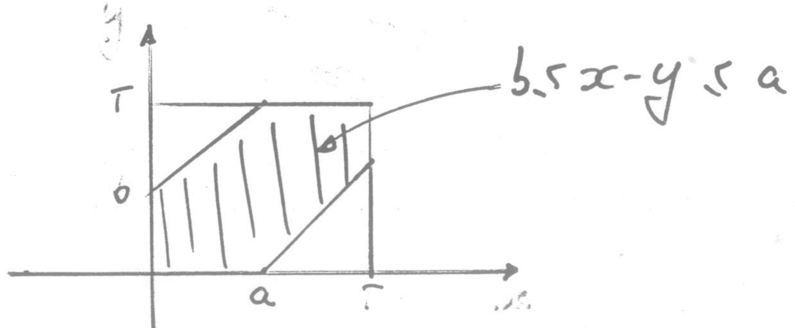
A: CONJUNTO
a_i: ELEMENTOS $\in A$
$a_{m+1} \notin A$

VAZIO: $\{\phi\}$

EXEMPLO: f_i face i do dado

$$S = \{f_1, f_2, \dots, f_6\} \quad m = 6$$

EXEMPLO:



$S: 0 \le x \le T$ e $0 \le y \le T$

ELEMENTOS: PARES $(x,y) \in S$

SUB: $b \le x-y \le a$

Operações

$A+B = A \cup B$

$B \subset A \Rightarrow A+B = A$

$A+A = A \qquad A+\{\phi\} = A \qquad S+A = S$

$AB = A \cap B$

$A \subset B \Rightarrow AB = A$

$AA = A \qquad \{\phi\}A = \{\phi\} \qquad AS = A$

Exemplo: $S = \{1, 2, 3, 4, 5, 6\}$

A : {par} $= \{2, 4, 6\}$

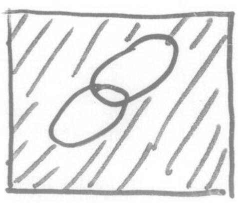
B : {menor que 5} $= \{1, 2, 3, 4\}$

$AB = \{2, 4\}$

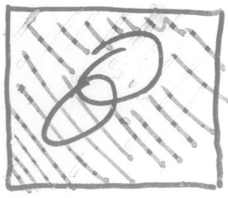
Lei de DeMorgan

$\overline{A+B} = \bar{A}\,\bar{B}$

$\overline{AB} = \bar{A} + \bar{B}$



$\overline{A+B}$



$\bar{A}\,\bar{B}$

AXIOMAS:

A cada evento A associa-se $P(A)$

- $P(A) \geq 0$
- $P(S) = 1$
- $P(A+B) = P(A) + P(B)$ para $AB = \{\phi\}$

$P(A+B) = P(A) + P(B) - P(AB)$

ESPAÇOS CONTÁVEIS

S com n resultados

$P(a_i) = p_i \quad , \sum_{i=1}^{n} p_i = 1$

Equiprobabilidade $\quad p_i = 1/n$

Exemplo: ~~Dado~~ MOEDA

$S = \{H, T\}$

EVENTOS $\{\phi\}$, $\{H\}$, $\{T\}$, S

$P\{H\} = p \qquad P\{T\} = q$

$p + q = 1$

Dado 3 vezes

HHH, HHT, HTH, THH, THT, TTH, HTT, TTT

$P\{HHH\} = 1/8$

$P\{H$ nas 2 1ª vezes$\} = P\{HHH, HHT\} = 2/8 = 1/4$

LINHA DO REAL

Nº infinito DE PONTOS

NÃO SE DEFINE PROB. P/ TODOS OS PTOS

$p(X = x_i) = 0$

Mas $P\{X \leq x\}$ OK!

PROBABILIDADE CONDICIONAL

$$P(A|B) = \frac{P(AB)}{P(B)}$$

EXEMPLO: DADO $P\{f_2|par\}$

$$P\{f_2|par\} = \frac{P\{f_2, par\}}{P\{par\}} = \frac{P\{f_2\}}{P\{par\}} = \frac{1/6}{3/6} = 1/3$$

EXEMPLO $\alpha(t) = 3.10^{-9}\, t^2 (100-t)^2 \quad 0 \le t \le 100$
arquivos de mortalidade

a) Prob. morrer entre 60 e 70

$$P\{60 \le t \le 70\} = \int_{60}^{70} \alpha(t)\, dt = 0.154$$

b) Prob. morrer entre 60 e 70
dado maior que 60.

$$P\{60 \le t \le 70 | t \geq 60\} = \frac{P\{60 \le t \le 70 \wedge t \geq 60\}}{P\{t \geq 60\}}$$

$$= \frac{\int_{60}^{70} \alpha(t)\, dt}{\int_{60}^{100} \alpha(t)\, dt} = 0.486$$

Teorema de Bayes

$$P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$$

Exemplo

Box	Nº Comp.	Defeituosos
1	2000	100
2	500	200
3	1000	100
4	1000	100

Retira-se 1

a) Prob. Defeituoso

$$P(D) = \sum_{i=1}^{4} P(D|B_i)\, P(B_i)$$

$$= \frac{1}{4} \times 0.05 + \frac{1}{4} \times 0.4 + \frac{1}{4} \times 0.1 + \frac{1}{4} \times 0.1$$

$$= 0.1635$$

b) $P(B2|D) = ? = \dfrac{P(B2 \cap D)}{P(D)} = \dfrac{P(D|B_2)\, P(B_2)}{P(D)} = \dfrac{0{,}4 \times 0.25}{0.1635}$

$$= 0.615$$

INDEPENDÊNCIA

A e B independentes

$$P(AB) = P(A)P(B)$$

TENTATIVAS REPETIDAS

TENTATIVAS BERNOULLI

Prob. EVENTO A OCORRER k vezes em n tentativas

$$P(A) = p \qquad P(\bar{A}) = 1 - p \triangleq q$$

ÚNICA ORDEM

$$p^k q^{n-k}$$

Qualquer ordem

$$p_n(k) = \binom{n}{k} p^k q^{n-k}$$

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

EXEMPLO : Dado 5 VEZES

$p_5(2) = ?$ $\binom{5}{2} p^2 (1-p)^3$, $p = 1/6$

$q = 5/6$

$= 16\%$

VARIÁVEIS ALEATÓRIAS VA

PROCESSO ASSOCIA UM NÚMERO $X(A)$ AO EVENTO A.

$\{X \leq x\}$ é um evento para cada x

$\{X = \infty\} = \{X = -\infty\} = \{\phi\}$

FUNÇÃO DIST. PROBABILIDADE

$$F_X(x) = P\{X \le x\} \qquad -\infty < x < \infty$$

$$F_X(\infty) = 1 \qquad F_X(-\infty) = 0$$

$$F_X(x_1) \le F_X(x_2) \Rightarrow x_1 \le x_2$$

$$P\{X > x\} = 1 - F_X(x)$$

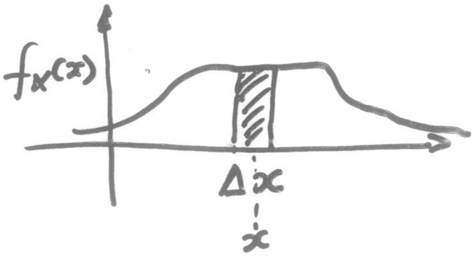
$$P\{x_1 < X \le x_2\} = F_X(x_2) - F_X(x_1)$$

Para VA Discretas

$$P\{X = x_i\} = p_i$$

FUNÇÃO DENS. PROBABILIDADE

$$f_X(x) = \frac{dF_X(x)}{dx}$$



$$f_X(x) \geq 0$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy \quad ; \quad F_X(\infty) = \int_{-\infty}^{\infty} f_X(y)\,dy = 1$$

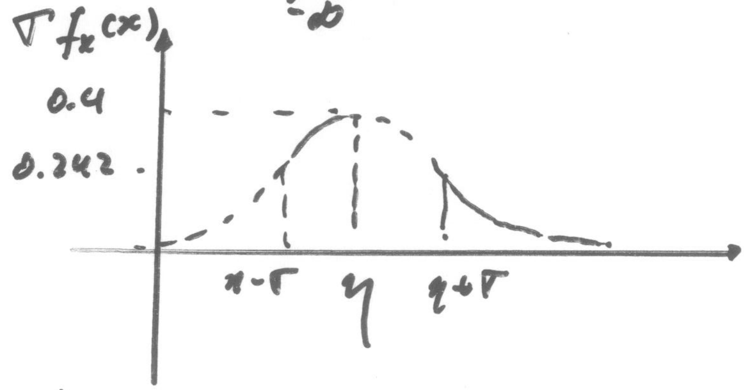
$$P\{x_1 < X \le x_2\} = \int_{x_1}^{x_2} f_X(x)\,dx$$

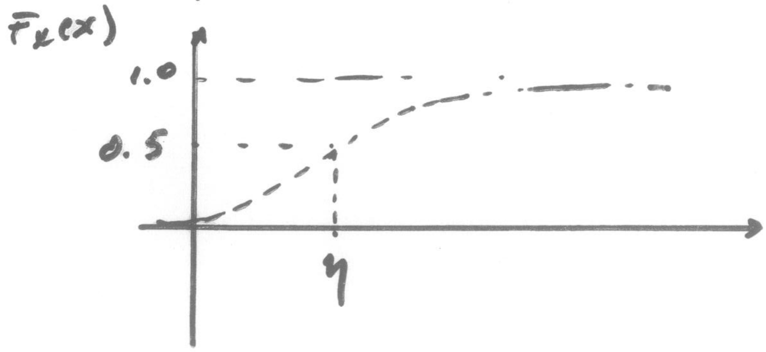
EXEMPLOS

NORMAL OU GAUSSIANA $N(\eta;\sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\left(\frac{x-\eta}{\sqrt{2}\,\sigma}\right)^2\right]$$

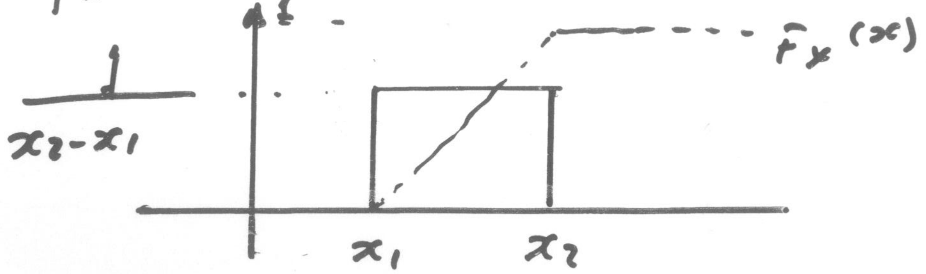
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$





UNIFORME

$$f_X(x) = (x_2 - x_1)^{-1} \qquad x_1 \le x \le x_2$$



$$F_X(x) = \int_{x_1}^{x} \frac{dx}{x_2 - x_1} = \frac{x}{x_2 - x_1}$$

BINOMIAL

$$P\{X=k\} = \binom{n}{k} p^k q^{n-k}$$

$$f_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \, \delta(x-k)$$

$$F_X(x) = \sum_{k=0}^{m} \binom{n}{k} p^k q^{n-k} \qquad m \le x < m+1$$

Poisson

$$P\{X=k\} = \frac{a^k}{k!}\exp(-a)$$

Gamma

$$f_X(x) = \frac{c^b}{\Gamma(b)} x^{b-1} \exp(-cx)$$

$$\Gamma(b+1) = \int_0^\infty y^b \exp(-y)\,dy$$

$$\Gamma(b+1) = b\,\Gamma(b)$$

$$b = n$$

$$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\Gamma(n-1) = \dots = n!$$

$$\Gamma(1/2) = \int_0^\infty y^{-1/2} e^{-y}\,dy = 2\int_0^\infty e^{-z^2}\,dz = \sqrt{\pi}$$

Erlang : n inteiro $(b=n)$

$$f_X(x) = \frac{x^{n-1}}{(x-1)!} c^n \exp(-cx)$$

Exponencial $n=1$

$$f_X(x) = c\exp(-cx)$$

Chi-Quadrada $b = n/2$ $c = 1/2$

$$f_X(x) = \frac{1}{2^{n/2}\,\Gamma(n/2)}\, x^{n/2-1}\, e^{-x}$$

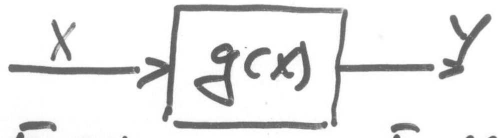
PROB. CONDICIONAIS

$$F_X(x) = \sum_{i=1}^{n} F_X(x \mid A_i)\, P(A_i)$$

$$f_X(x) = \sum_{i=1}^{n} f_X(x \mid A_i)\, P(A_i)$$

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X = x)\, f_X(x)\, dx$$

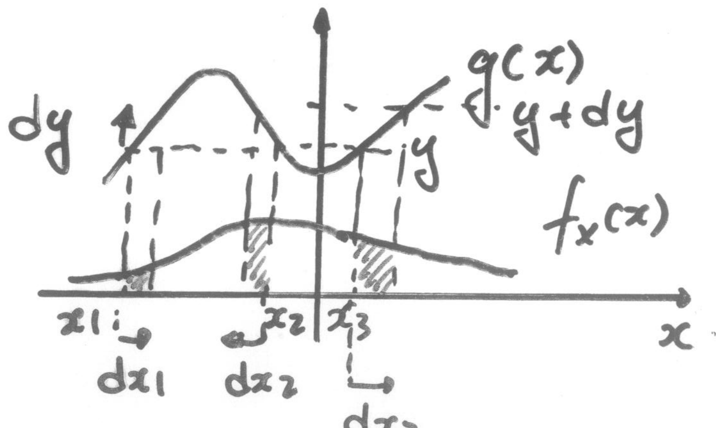
TRANSFORMAÇÃO DE VARIÁVEIS



$F_X(x)$ $\qquad$ $F_Y(y) = ?$

$f_X(x)$ $\qquad$ $f_Y(y) = ?$

Tensão → Potência
→ Retificada
→ Quantizada
→ Limitada



$$f_Y(y)|dy| = \sum_{i=1}^{n} f_X(x)|dx|\Big|_{x=x_i}$$

$$\boxed{f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x)}{\left|\frac{dg(x)}{dx}\right|}\Bigg|_{x=x_i}}$$

$$f_Y(y|A) = \sum_{i=1}^{n} \left. \frac{f_X(x|A)}{\left|\frac{dg(x)}{dx}\right|} \right|_{x=x_i}$$

Exemplos:

1) $V = 0.01r + 10$

r uniforme entre 900 e $1100 r$

$$\left|\frac{dv}{dr}\right| = 0.01$$

$$f_V(v) = \frac{f_R(r)}{0.01} = \frac{1}{200 \times 0.01} = \frac{1}{2} \quad 19 \leq v \leq 21$$

2) $Y = 1/X$

$$|dy/dx| = 1/x^2 = 1/y^2$$

$$f_Y(y) = \frac{1}{y^2} f_X(x) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right)$$

3. $Y = aX^2$

$y = ax^2 \quad ; \quad |dy/dx| = 2ax$

$x_1 = \sqrt{y/a} \qquad x_2 = -\sqrt{y/a}$

$$f_Y(y) = \frac{1}{2a\sqrt{y/a}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right]$$

4. $Y = \sqrt{X} \Rightarrow x = y^2$

$|dy/dx| = x^{-1/2}/2$

$f_Y(y) = 2y\, f_X(y^2) \qquad y > 0$

5. $Y = e^X \qquad y = e^x \Rightarrow x = \ln y$

$|dy/dx| = e^x = y$

$\therefore f_Y(y) = \frac{1}{y} f_X(\ln y)$

Seja $X \sim N(\mu; \sigma)$

$$f_Y(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{\ln y - \mu}{\sqrt{2}\sigma}\right)^2\right]$$

6. $Y = a\sin(X+\theta)$

$$y = a\sin(x+\theta)$$

$$|dy/dx| = a\cos(x+\theta) = \sqrt{a^2-y^2}$$

$$f_Y(y) = \frac{1}{\sqrt{a^2-y^2}} \sum_{n=-\infty}^{\infty} f_X(x_n) \qquad |y| < a$$

X uniforme $(-\pi, \pi)$

$$f_Y(y) = \frac{2}{2\pi\sqrt{a^2-y^2}} = \frac{1}{\pi\sqrt{a^2-y^2}}$$

MÉDIA ESPERANÇA VALOR ESPERADO

$$E\{X\} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \triangleq \eta$$

X Uniforme (x_1, x_2)

$$E\{X\} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} x\, dx = \frac{x_1 + x_2}{2}$$

Dist. Discreta

$$f_X(x) = \sum_i p_i\, \delta(x - x_i)$$

$$E\{X\} = \int_{-\infty}^{\infty} x \sum_i p_i\, \delta(x - x_i)\, dx$$

$$= \sum_i p_i \int_{-\infty}^{\infty} x\, \delta(x - x_i)\, dx$$

$$\boxed{E\{X\} = \sum_i x_i p_i}$$

Exemplo: $X = \{1, 2, \dots, 6\}$ $p_i = 1/6$

$$E\{X\} = \{1 + 2 + \dots + 6\}\frac{1}{6} = 3.5$$

$$E\{X|B\} = \int_{-\infty}^{\infty} x f_X(x|B)\, dx$$

$$E\{X|B\} = \sum_i x_i P\{X = x_i|B\}.$$

VARIÂNCIA

$$\sigma^2 = E\{(X-\bar{X})^2\} = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\, dx$$

$$\sigma^2 = E\{(X-\bar{X})^2\} = E\{X^2 - 2X\bar{X} + \bar{X}^2\}$$

$$= E\{X^2\} - 2\bar{X} E\{X\} + E\{\bar{X}^2\}$$

$$= E\{X^2\} - 2\bar{X}^2 + \bar{X}^2$$

$$\boxed{\sigma^2 = E\{X^2\} - E^2\{X\}}$$

MOMENTOS

$$m_n = E\{X^n\} = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

$$\mu_n = E\{(X-\eta)^n\} = \int_{-\infty}^{\infty} (x-\eta)^2 f_X(x)\,dx$$

$$E\{|X|^n\} = \int_{-\infty}^{\infty} |x|^n f_X(x)\,dx$$

$$E\{|(X-a)^n\} = \int_{-\infty}^{\infty} |x-a|^n f_X(x)\,dx$$

CONDICIONAIS

$$f_X(x|A) = \frac{d}{dx} F_X(x|A)$$

$$E[X|A] = \int_{-\infty}^{\infty} x\, f_X(x|A)\,dx$$

$$E[X|X \geq a\} = \int_{-\infty}^{\infty} x\, f_X(x|X \geq a)\,dx$$

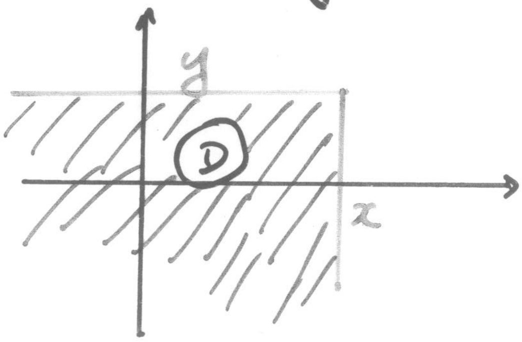
VA $N(0;\sigma)$

$$E\{X^m\} = \begin{cases} 0 & m = 2k+1 \\ 1\cdot 3 \cdots (m-1)\,\sigma^m & m = 2k \end{cases}$$

$$E\{|X|^m\} = \begin{cases} 1\cdot 3 \cdots (m-1)\,\sigma^m & m = 2k \\ 2^k\, k!\, \sigma^{2k+1}\sqrt{2/\pi} & m = 2k+1 \end{cases}$$

DUAS VARIÁVEIS ALEATÓRIAS

$$\{X \le x, Y \le y\} = \{(X,Y) \in D\}$$



$$F_{XY}(x,y) = P\{X \le x, Y \le y\}$$

- $F_{XY}(-\infty, y) = 0$ $\quad F_{XY}(x, -\infty) = 0$
- $F_{XY}(x, \infty) = F_X(x)$ $\quad F_{XY}(\infty, y) = F_Y(y)$
- $F_{XY}(-\infty, -\infty) = 0$ $\quad F_{XY}(\infty, \infty) = 1$

$$f_{XY}(x,y) = \partial^2 F_{XY}(x,y) / \partial x \partial y$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\alpha, \beta)\, d\alpha\, d\beta$$

$$P\{(X,Y) \in D\} = \iint_D f_{XY}(x,y)\, dx\, dy$$

ESTATÍSTICAS MARGINAIS

$$F_X(x) = F_{XY}(x, \infty) \qquad F_Y(y) = F_{XY}(\infty, y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy \qquad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

INDEPENDÊNCIA: X e Y INDEPENDENTES

$$F_{XY}(x,y) = F_X(x)\, F_Y(y)$$

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

TRANSFORMAÇÃO DE VAs

Dados $f_{XY}(x,y)$

$z = g(x,y)$, $h(x,y) = w$

$f_{ZW}(z,w) = ?$

$$\boxed{f_{ZW}(z,w) = \sum_{i=1}^{n} \left. \frac{f_{XY}(x,y)}{|J(x,y)|} \right|_{x_i, y_i}}$$

$$J(x,y) = \begin{vmatrix} \partial z/\partial x & \partial z/\partial y \\ \partial w/\partial x & \partial w/\partial y \end{vmatrix}$$

$$= \begin{vmatrix} \partial x/\partial z & \partial x/\partial w \\ \partial y/\partial z & \partial y/\partial w \end{vmatrix}^{-1}$$

JACOBIANO

Momentos Conjuntos

$$E\{g(X,Y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy$$

$$E\{g(X,Y)\} = \sum_{i,k} g(x_i, y_k)\, p_{ik}$$

Covariância

$$C = E\{(X-\eta_x)(Y-\eta_y)\}$$

$$C = E\{XY\} - E\{X\}E\{Y\}$$

$$|C| \leq \sigma_x \sigma_y$$

Correlação

$$r = C / \sigma_x \sigma_y$$

$$|r| \leq 1$$

Descorrelação

$$C = 0 \;,\; r = 0 \;,\; E\{XY\} = E\{X\}E\{Y\}$$

Ortogonalidade

$$E\{XY\} = 0$$

Momentos

$$m_{kr} = E\{X^k Y^r\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^k y^r f_{XY}(x,y)\, dx\, dy$$

$$m_{10} = \eta_x \qquad m_{01} = \eta_y$$

$$m_{20} = E\{X^2\} \qquad m_{20} = E\{Y^2\}$$

Momento Central

$$\mu_{kr} = E\{(X-\eta_x)^k (Y-\eta_Y)^r\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\eta_x)^k (y-\eta_y)^r f_{XY}(x,y)\, dx\, dy$$

$$\mu_{10} = \mu_{01} = 0$$

$$\mu_{11} = C, \quad \mu_{20} = \sigma_x^2 \quad \mu_{02} = \sigma_Y^2$$

Teorema de Bayes

$$f_X(x|y) = \frac{f_X(y|x)\, f_X(x)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

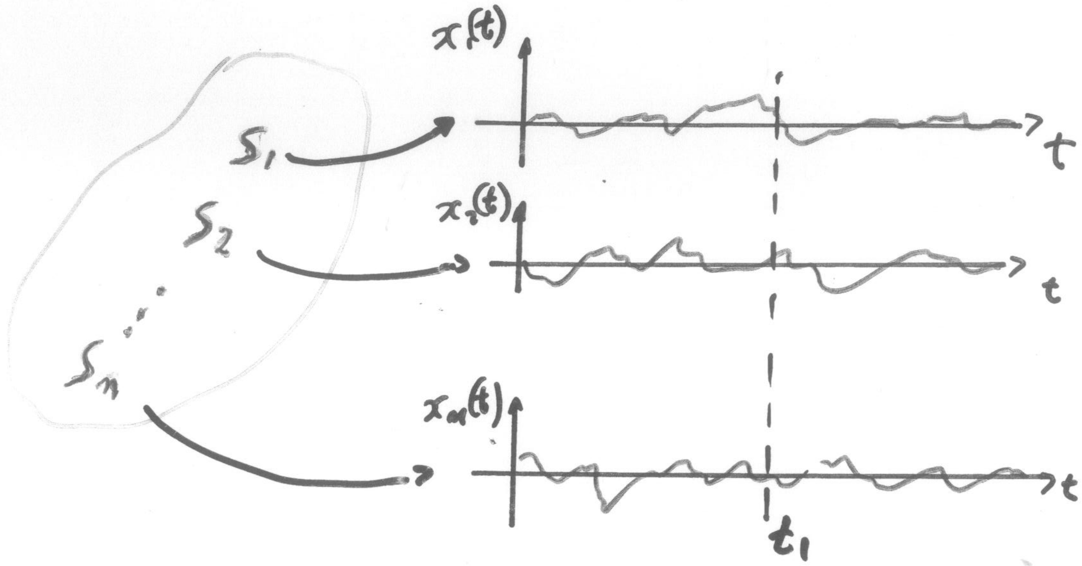
$$f_{XY}(x,y) = f_Y(y|x)\, f_X(x)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_Y(y|x)\, f_X(x)\, dx$$

$$f_X(x|y) = \frac{f_Y(y|x)\, f_X(x)}{\int_{-\infty}^{\infty} f_Y(y|x)\, f_X(x)\, dx}$$

PROCESSOS ESTOCÁSTICOS

CONJUNTO DE FUNÇÕES TEMPORAIS COM UMA REGRA PROBABILÍSTICA. ASSOCIA UMA PROBABILIDADE A QUALQUER EVENTO ASSOCIADO A UMA OBSERVAÇÃO DESTAS FUNÇÕES



$X(t)$ é um processo constituído das funções

$\{x_j(t)\}$, $j = 1, 2, \ldots, n$

$x_1(t)$ tem prob. $P\{S_1\}$ e corresponde ao

pto amostral do espaço S.

A coleção $\{x_j(t = t_1)\}$ $j = 1, 2, \ldots, n$ forma

uma VA $X(t_1)$

Num Proc. Aleatório a VA é função do tempo

- Estacionaridade

Sentido estrito

$$f_{X(t)}(x) = f_{X(t+T)}(x)$$

vetores

As componentes de $X(t)$ são observadas em $t_1, t_2, \ldots, t_m$

As componentes de $X(t+T)$ são obs. em $t_1+T, t_2+T, \ldots, t_m+T$

Média

$$m_x(t_k) = E\{X(t_k)\} \quad \forall t_k$$
$$= \int_{-\infty}^{\infty} x \, f_{X(t_k)}(x) \, dx$$

Autocorrelação

$$R_X(t_k, t_i) = E\{X(t_k) X(t_i)\}$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x y \, f_{X(t_k)} f_{X(t_i)} \, dx \, dy$$

AUTOCOVARIÂNCIA

$$K_X(t_k, t_i) = E[(X(t_k) - m_X(t_k))(X(t_i) - m_X(t_i))]$$

$$= R_X(t_k, t_i) - m_X(t_k)\, m_X(t_i)$$

Sentido Amplo

$$m_X(t_k) = m_X \qquad \forall\, t_k$$

$$R_X(t_k, t_i) = R_X(t_k - t_i)$$

$$K_X(t_k, t_i) = K_X(t_k - t_i)$$

Para $\tau = t_k - t_i$

$$R_X(\tau) = E[X(t)\,X(t-\tau)] = E[X(t+\tau)\,X(t)]$$

Note que

$$R_X(\tau)(0) = E[X^2(t)]$$

$$R_X(\tau) = R_X(-\tau)$$

$$|R_X(\tau)| = R_X(0)$$

Wiener-Khintchine

Densidade Espectral de Potência

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau)\, d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(j2\pi f\tau)\, df$$

Note que

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df$$

$$S_X(f) \geq 0$$

$$S_X(f) = S_X(-f)$$

ERGODICIDADE

Dificuldade em caracterizar a fdp em processos reais. Usa-se média temporal

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

$$\langle x(t)\, x(t-\tau) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t-\tau)\, dt$$

MÉDIA TEMPORAL = MÉDIA CONJUNTO

$$E\left[\frac{1}{2T} \int_{-T}^{T} x(t)\, dt\right] = \frac{1}{2T} \int_{-T}^{T} E[x(t)]\, dt =$$

$$= \frac{1}{2T} \int_{T}^{T} m_x\, dt = m_x$$

$$E\left[\frac{1}{2T} \int_{T}^{T} x(t)\, x(t-\tau)\, dt\right] = \frac{1}{2T} \int_{-T}^{T} E[x(t)\, x(t-\tau)]\, dt$$

$$= R_x(\tau)$$

UM PROCESSO ALEATÓRIO É

ERGÓTICO NA MÉDIA SE

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt = m_x$$

com prob = 1.

Teorema do Limite Central

The probability distribution function of a sum of independent RV approaches that of a Gaussian RV as the number of independent RVs increases without limit.

Spread Spectrum

- **Spread Spectrum is defined as a communication technique in which the intend signal is spread over a bandwidth in excess of the minimum bandwidth required to transmit the signal.**
  - This is accomplished by the use of a wideband encoding signal at the transmitter, which operates in synchronism with the receiver, where the encoding signal is also known.
  - Generating a spread spectrum signal involves two steps:
    - first, the carrier is modulated by the baseband digital information with rate $R_b = 1/T_b$
    - second, the modulated signal is used to modulate a wideband funtion with rate $R_c = 1/T_c$.

Spread Spectrum

- The desired wideband signal arrives at the receiver together with other wideband signals, interference, and noise.
- Other waveforms are not correlated and will be spread, appearing a noise to the modulator.
- The correlated signal is then a band-pass signal, whereas the noise component is a wideband signal.
- Two main spread spectrum techniques are used:
  - direct sequence spread sequence spread spectrum;
  - and frequency hopping spread spectrum.

Correlation

- **The correlation function quantifies the degree of similarity between two functions.**
  - Lets $x(t)$ and $y(t)$ be two nonperiodic waveforms with finite energy. The cross-correlation function $R_{x,y}(\tau)$ is given by

$$R_{x,y}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt$$

  - If $x(t)$ and $y(t)$ are periodic waveforms, with period T, then

$$R_{x,y}(\tau) = \frac{1}{T}\int_{0}^{T} x(t)y(t-\tau)dt$$

  - The autocorrelation $R_z(\tau)$ for either type of waveform $z(t)$ is defined as

$$R_z(\tau) = R_{z,z}(\tau) = \int_{-\infty}^{\infty} z(t)z(t-\tau)dt$$

Correlation

- Assume that *z(t)* is a binary waveform defined as

$$z(t) = \sum_{k=-\infty}^{\infty} Z_k \hat{Z}\left(t - \frac{k}{W}\right)$$

where $Z_k \in \{+1,-1\}$, is the pulse shape, and $1/W$ is the duration of the pulse. Then

$$R_{x,y}(\tau) = \frac{W}{K} \sum_{k=-\infty}^{\infty} R_{X,Y}(k) R_{\hat{X},\hat{Y}}\left(\tau - \frac{k}{W}\right)$$

where K is the number of pulses composing the period of the sequence, $R_{X,Y}(k)$ is the cross-correlation function of the two periodic binary sequences X_k and Y_k, and $R_{\hat{X},\hat{Y}}(\tau)$ is the nonperiodic cross-correlation function for the basic waveforms $\hat{X}(t)$.. and $\hat{Y}(t)$. The cross-correlation between X_k and Y_k is defined as

$$R_{X,Y}(k) = \sum_{i=0}^{K-1} X_i Y_{i-k}$$

Correlation

- The autocorrelation function $R_Z(k)$ of the sequence Z_i is defined as

$$R_Z(k) = R_{Z,Z}(k) = \sum_{i=0}^{K-1} Z_i Z_{i-k}$$

- The autocorrelation function for $z(t)$ is

$$R_z(\tau) = R_{z,z}(\tau) = \frac{W}{K} \sum_{k=-\infty}^{\infty} R_Z(k) R_{\hat{Z}}\left(\tau - \frac{k}{W}\right)$$

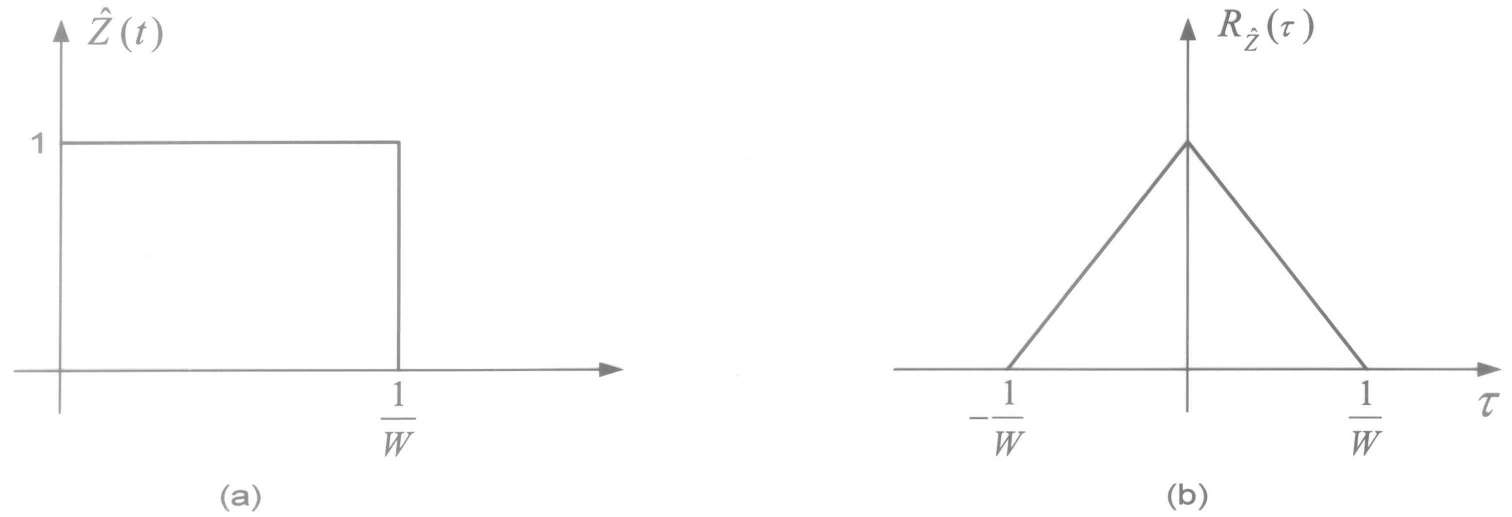


Figure C.1
A rectangular pulse (a) and its autocorrelation (b).

Correlation

- Now, for a sequence of *K* binary symbols, in which the number of *+1s* and *-1s* differ by one, the autocorrelation is *K*, for *k=0* and *-1* for *k≠0*.

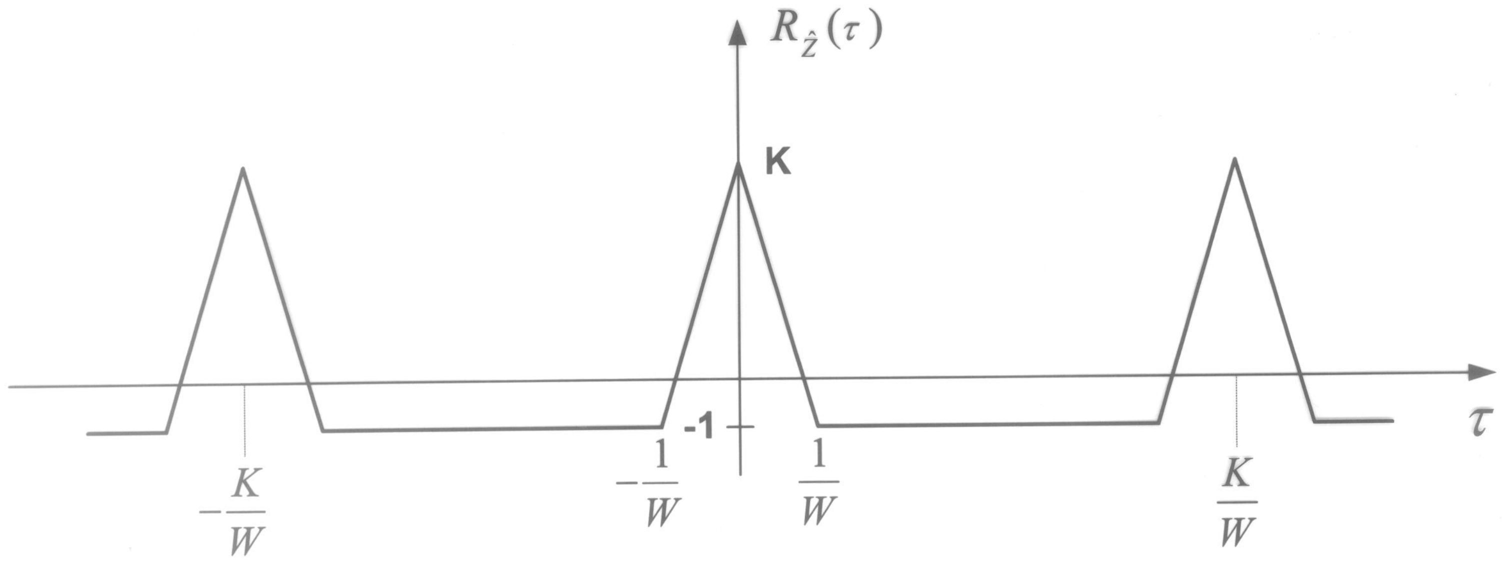


Figure C.2
Autocorrelation function of a real signal waveform

- Two real-valued waveforms are said to be orthogonal if $R_{x,y}(0)=0$.

Pseudonoise Sequences

- **Pseudonoise (PN) or pseudorandom sequences are used for two main purposes data scrambling and spread spectrum modulation.**
- Note, in the scrambling operation (as well as in the modulation operation), that both transmitter and receiver must work exactly the same PN sequence.
- A sequence with a period equal to 2^n-1 is known as maximal length sequence or *m*-sequence or PN sequence. The following main properties characterize the *m*-sequence:
    - *Balance Property.* Within a complete period of sequence, the number of 1s and 0s differs from each other by at most 1.
    - *Correlation Property.* By comparing a complete sequence with any shifted version of it, within the sequence period, the number of agreements minus the number of disagreements is always -1.

Walsh Codes

- **The Walsh sequence can be generated by means of Rademacher functions or by the Hadamard matrices.**
  - The Hadamard matrix is defined as

$$H_0 = [1]$$

$$H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

  - The Walsh sequences are indexed by the row of matrix. An example of the Hadamard matrix for *n*=2 is shown as follows:

$$H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Orthogonal Variable Spreading Factor Codes

- Channelization in multirate CDMA systems can be provided by orthogonal variable spreading factor (OVSF) codes.
- Uplink and downlink channels make use of OVSF codes.
- The OVSF codes preserve the orthogonality between channels of different rates and spreading factors.
- They can be defined as

$$C_{1,0} = [1]$$

$$\begin{bmatrix} C_{2,0} \\ C_{2,1} \end{bmatrix} = \begin{bmatrix} C_{1,0} & C_{1,0} \\ C_{1,0} & -C_{1,0} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Orthogonal Variable Spreading Factor Codes

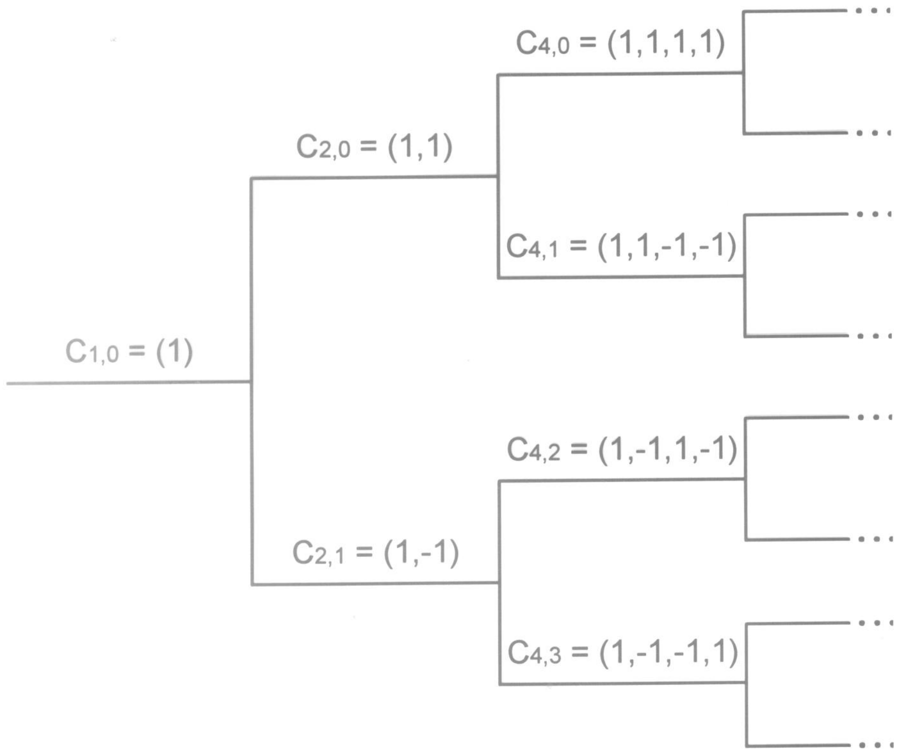


Figure C.3
OVSF code tree.

$$\begin{bmatrix} C_{2^{n+!},0} \\ C_{2^{n+!},1} \\ C_{2^{n+!},2} \\ C_{2^{n+!},3} \\ \vdots \\ C_{2^{n+!},2^{n+1}-2} \\ C_{2^{n+!},2^{n+1}-1} \end{bmatrix} = \begin{bmatrix} C_{2^n,0} & C_{2^n,0} \\ C_{2^n,0} & -C_{2^n,0} \\ C_{2^n,1} & C_{2^n,1} \\ C_{2^n,1} & -C_{2^n,1} \\ \vdots & \vdots \\ C_{2^n,2^n-1} & C_{2^n,2^n-1} \\ C_{2^n,2^n-1} & -C_{2^n,2^n-1} \end{bmatrix}$$

Rake Receiver

- In a multipath propagation environment, the receiverd signal contains replicas of attenuated and delayed version of the transmitted signal.
- Assume that the signal is pseudorandom with a correlation width of $1/W$.

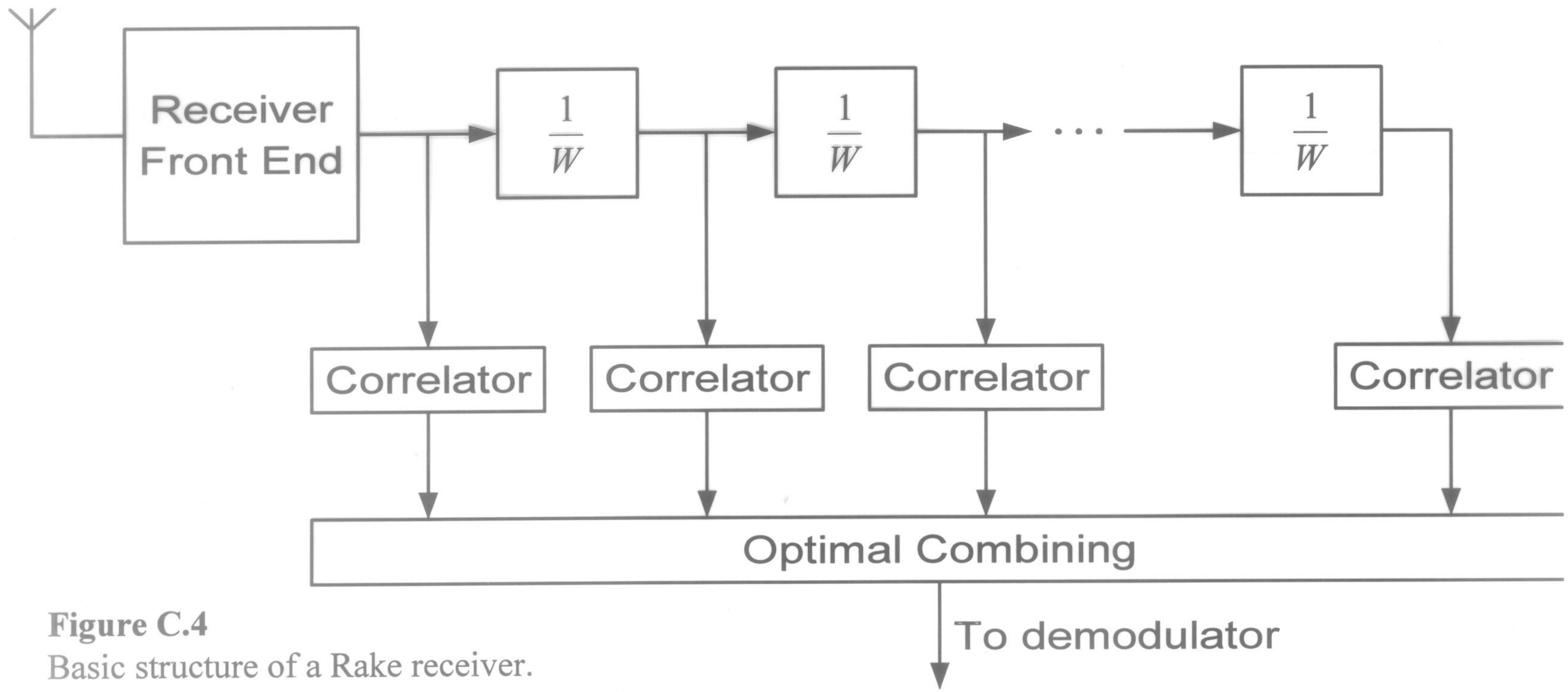


Figure C.4
Basic structure of a Rake receiver.

Processing Gain

- **Processing gain *G* is defined as the ratio between the bandwidth *W* of the spread signal and the bandwidth *w* of the unspread signal, i.e.,**

$$G = \frac{W}{w}$$

which represents the gain achieved by processing a spread spectrum signal over an unspread signal.

- It can be obtained by the difference in decibels between the output signal-to-noise ratio (SNR_o, SNR of the spread information) and the input signal-to-noise ratio (SNR_i, SNR of unspread information), i.e.,

$$10\log G = SNR_o - SNR_i$$

Direct Sequence Spread Spectrum

- **Direct sequence (DS) spread spectrum (SS) uses PN sequence to modulate a carrier.**
- In principle, any modulation technique such as AM (pulse), FM, or PM can be used. However, the most widespread form is the binary phase shift keying (BPSK) modulation.
- At the receiver, which is assumed to operate in synchronism with the transmitter, an exact replica of PN codes is used to unspread the received signal.

Direct Sequence Spread Spectrum

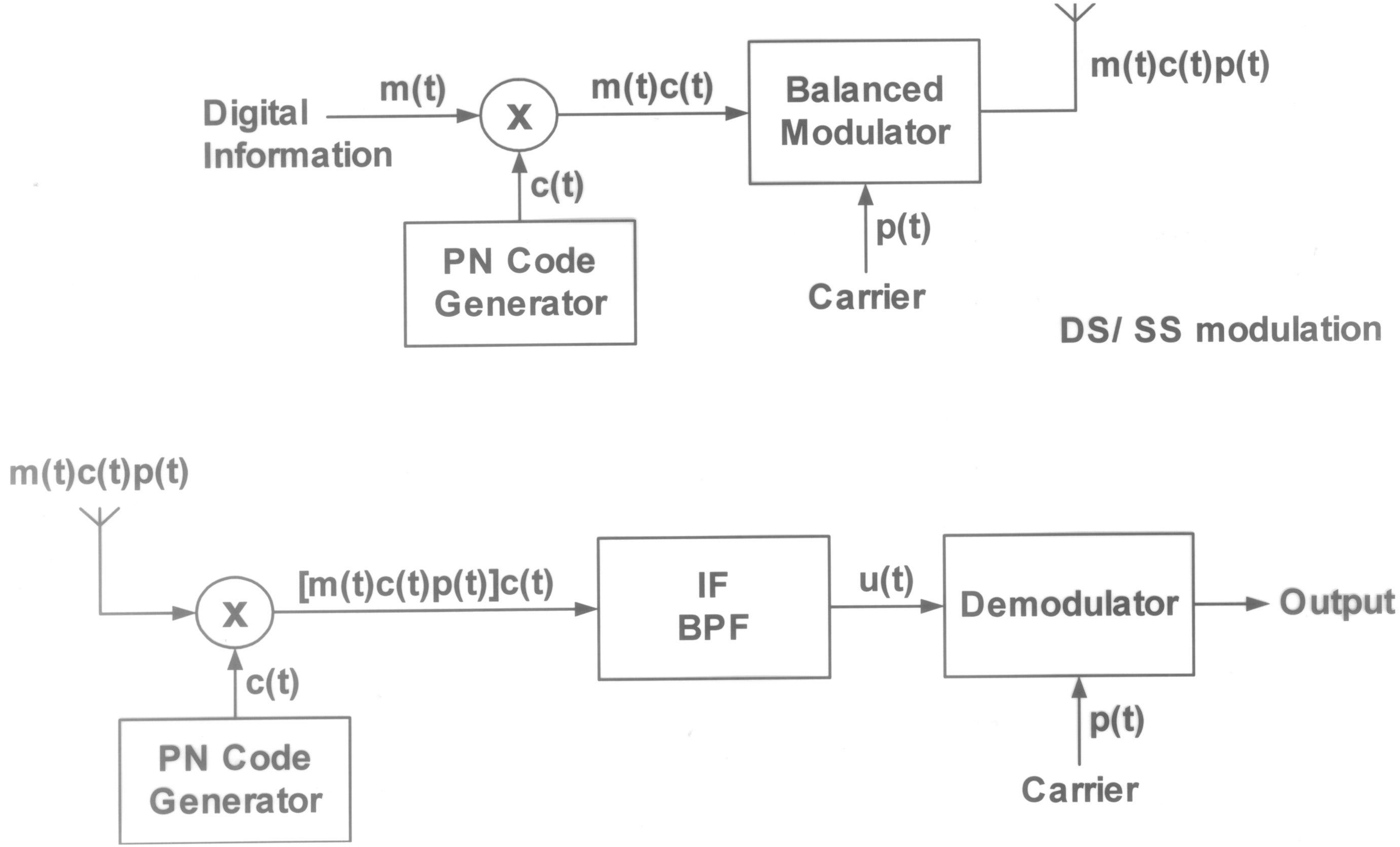


Figure C.5
.A simplified model of DS/SS system.

Frequency Hopping Spread Spectrum

- **Frequency hopping (FH) is a spread spectrum (SS) technique in which the carrier is allowed to hop from one frequency to another in a sequence dictated by a PN code.**
- At the receiver, which is assumed to operate in synchronism with the transmitter, the signal is mixed with a locally generated replica of the transmitter frequency sequence, offset by the intermediate frequency f_{IF}.
- The two basic FH systems: slow FH (SFH) and fast FH (FFH).
  - In SFH systems, several symbols of information are transmitted on each frequency hop, where each symbol is a chip.
  - In FFH, several hops occur during the transmission of one symbol, where the chip is characterized by hop.

Frequency Hopping Spread Spectrum

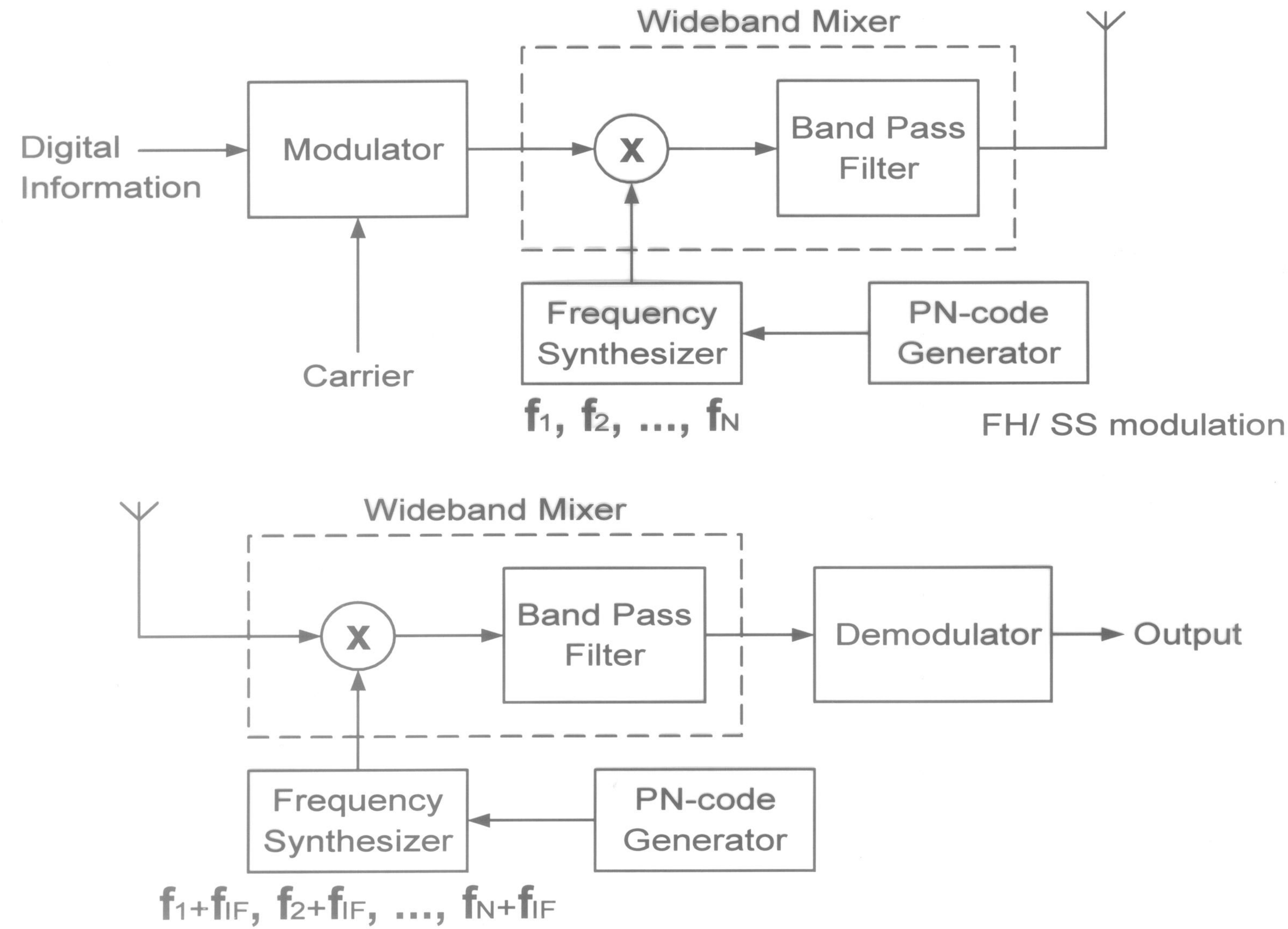


Figure C.5
.A simplified model of FH/SS system.

TRAFFIC ASPECTS IN MOBILE RADIO SYSTEMS

12.1 INTRODUCTION

Traffic Characteristics in Mobile Radio Systems is Distinct

Erlang-B is widely used

Given traffic and cochannel requirements, region is divided into cells

Given traffic per cell and blocking, number of channels are determined

Same size cells but different number of channels

Same number of channels but size cells

Traffic Distribution Vary in Time and in Space

Rush hour: Bell Shaped

Happy hour: Upside Down Bell

Handoffs, Roaming

12.2 QUEUEING AND TRAFFIC THEORY FUNDAMENTALS

Calls arrival are Poisson distributed

Holding times have a negative exponential distribution

Blocked calls are lost

12.2.1 Call Arrival Process

Call arrival: random process

Divide t in n t/n subintervals (n large)

(i) only one arrival can occur in any subinterval t/n

(ii) call arrivals are independent from each other

(iii) the probability $p_1(1)$ that an arrival occurs in one of the subintervals is proportional to the subinterval length. Hence $p_1(1) = \lambda t/n$, where $\lambda > 0$. Accordingly, the probability of no arrivals in t/n is $1 - \lambda t/n$.

The probability of exactly k arrivals in n subintervals can be evaluated using the binomial distribution. Then

$$p_k(n) = \binom{n}{k}\left(\lambda\frac{t}{n}\right)^k\left(1 - \lambda\frac{t}{n}\right)^{n-k} \qquad (12.1)$$

As n goes to infinity

$$p_k = \frac{(\lambda t)^k}{k!}\exp(-\lambda t) \qquad (12.2)$$

Mean: $E[k] = \sum_{k=0}^{\infty} kp_k = \lambda t$ (λ is the mean arrival rate of calls)

Variance: $E[k^2] - E^2[k] = \lambda t$

Mean Interarrival Time

τ: time between adjacent arrivals

$$A(t) = prob(\tau \le t) = 1 - prob(\tau > t)$$

But $prob(\tau > t)$ = probability that no arrivals occur within t (p_0)

Then, with $k = 0$ in Equation 12.2

$$A(t) = 1 - \exp(-\lambda t) \qquad (12.3)$$

The density a(t) is obtained by differentiating A(t) with respect to t

$$a(t) = \lambda\exp(-\lambda t) \qquad (12.4)$$

Equation (12.4) is referred to as the negative exponential distribution

$$E[t] = \int_0^{\infty} ta(t)dt = 1/\lambda$$

Memoryless Property of the Negative Exponential Distribution

Past history of an exponentially distributed random variable has no influence in predicting its future (the time until next call, given arrived call has just arrived is also negative exponential distributed)

Let t_0 be a time where no arrival occurs. What is the probability that the next arrival occurs within a time t from t_0?

$$prob(\tau \le t + t_0 \mid \tau > t_0) = ?$$

$$\text{prob}(\tau \le t + t_0 \mid \tau > t_0) = \frac{\text{prob}(t_0 < \tau \le t + t_0)}{\text{prob}(\tau > t_0)}$$

$$= \frac{\text{prob}(\tau \le t + t_0) - \text{prob}(\tau \le t_0)}{\text{prob}(\tau > t_0)} \qquad (12.5)$$

But prob($\tau \le x$) is given by A(x). Therefore

$\text{prob}(\tau \le t + t_0 \mid \tau > t_0) = 1 - \exp(-\lambda t)$

12.2.2 Call Holding Time

Divide t in n t/n subintervals (n large)

(i) The probability that a call terminates within one subinterval is proportional to its length. That is, this probability equals $\mu t/n$, where $\mu > 0$

(ii) The call termination occurs independently of which subinterval is considered.

Define:

τ: call holding time

H(t), h(t): distribution and density of τ

Then

$$1 - H(t) = 1 - \text{prob}(\tau \le t) = \text{prob}(\tau > t)$$

The probability prob($\tau > t$): prob. a call originated at time zero will not terminate before t is given by the probability that it will not terminate in any of the n subintervals of length t/n, when n tends to infinity. Then

$$1 - H(t) = \lim_{n \to \infty} \left(1 - \mu \frac{t}{n}\right)^n = \exp(-\mu t)$$

Thus

$$H(t) = 1 - \exp(-\mu t) \qquad (12.6)$$

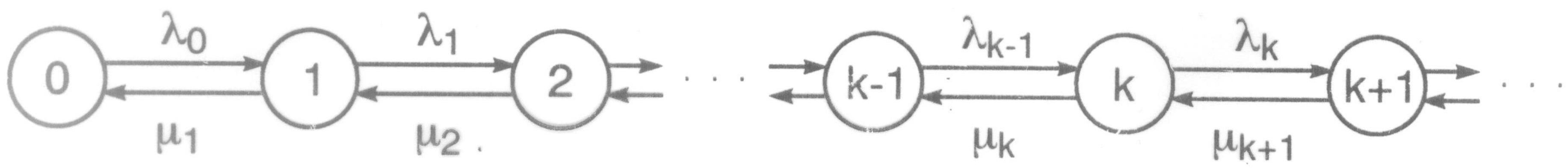


FIG12-1 200%

$$h(t) = \mu \exp(-\mu t) \qquad (12.7)$$

$E[t] = 1/\mu$

12.2.3 Birth-Death Processes

"A Markov process with a discrete state space is referred to as a Markov chain. A set of random variables $\{X_n\}$ forms a Markov chain if the probability that the next value (state) is x_{n+1} depends only upon the current value (state) x_n and not upon any previous values"

$$\text{prob}\left[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n, \ldots, X(t_1) = x_1\right] = \text{prob}\left[X(t_{n+1}) = x_{n+1} \mid X(t_n) = x_n\right] \qquad (12.8)$$

Birth-Death Process: transitions occur only between neighbouring states

S_k: system state when population is k

From S_k to S_{k+1}: a birth, with rate λ_k

From S_k to S_{k-1}: a death, with a rate μ_k

λ_k and μ_k are rates; $\lambda_k dt$, $\mu_k dt$ are probabilities

One-Dimensional Birth-Death Process

$p_k(t)$: prob. to be in S_k at the time instant t

prob. of reaching the state $S_k = \left[\lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t)\right]dt$

prob. of departing from the state $S_k = (\lambda_k + \mu_k)dtp_k(t)$

Difference between these above probabilities equals $dp_k(t)$

$$\frac{dp_k(t)}{dt} = \lambda_{k-1}p_{k-1}(t) + \mu_{k+1}p_{k+1}(t) - (\lambda_k + \mu_k)p_k(t) \qquad (12.9)$$

$t \rightarrow$ to infinity, system tends to equilibrium, $dp_k(t)/dt \rightarrow$ zero

$p_k(t) = p_k$

$$\lambda_{k-1}p_{k-1} + \mu_{k+1}p_{k+1} = (\lambda_k + \mu_k)p_k \qquad , k \geq 0 \tag{12.10}$$

$k = 0,1,2,3,\ldots,$ We must have

$$\sum_{k=0}^{\infty} p_k = 1 \tag{12.11}$$

We may solve the set of equations to obtain

$$p_k = p_0 \prod_{i=0}^{k-1} \lambda_i/\mu_{i+1} \tag{12.12}$$

where

$$p_0 = \left[1 + \sum_{k=1}^{\infty} \prod_{i=0}^{k-1} \lambda_i/\mu_{i+1}\right]^{-1} \tag{12.13}$$

Blocked Calls Held

System with an infinite number of servers

Birth rate is constant (λ)

Death rate rate from each server is constant (μ)

$$\lambda_k = \lambda \quad , \qquad k \geq 0$$

$$\mu_k = k\mu \quad , \qquad k \geq 1$$

Then

$$p_k = \left[(\lambda/\mu)^k/k!\right]\left[\sum_{i=0}^{\infty} (\lambda/\mu)^i\right]^{-1}$$

Hence

$$p_k = \frac{A^k}{k!}\exp(-A) \tag{12.14}$$

where $A = \lambda/\mu$ is the traffic offered in Erlangs (erl.)

Mean and Variance = A

Assume: system with N servers

Arrivals after N busy remain in system

When channel becomes free, customer is served

$$B = \sum_{k=N}^{\infty} \frac{A^k}{k!} \exp(-A) \tag{12.15}$$

The above equation is referred to as Molina's formula

Blocked Calls Cleared (Finite States)

System with N channels

Calls arriving when all the channels are found to be busy are lost

$$\lambda_k = \lambda \quad , \qquad k \leq N - 1$$

$$\mu_k = k\mu \quad , \qquad k < N$$

$$p_k = \frac{A^k/k!}{\sum_{i=0}^{N} A^i/i!} \tag{12.15}$$

Blocking will occur when all the N channels are busy

$$E(A,N) = p_N = \frac{A^N/N!}{\sum_{i=0}^{N} A^i/i!} \tag{12.16a}$$

which is known as Erlang-B formula

Recursive form

$$E(A,N) = \frac{E(A,N-1)}{N/A + E(A,N-1)} \tag{12.16b}$$

Approximated form (N large)

$$E(A,N) \simeq \frac{A^N}{N!} \exp(-A) \tag{12.17}$$

Blocked Calls Delayed

Blocked calls are allowed to queue up and wait to be served

$$\lambda_k = \lambda \quad , \qquad k \geq 0$$

$$\mu_k = \begin{cases} k\mu \ , & 0 \leq k \leq N \\ N\mu \ , & k \geq N \end{cases}$$

Using the same procedure as in the previous case we obtain

$$p_k = \frac{A^k}{k!} p_0 \quad , \qquad 0 \leq k \leq N$$

$$p_k = \frac{A^k}{k!} N^{N-k} p_0, \qquad k \geq N$$

where

$$p_0 = \left[\sum_{k=0}^{N-1} \frac{A^k}{k!} + \frac{A^N}{N!} \frac{1}{1 - A/N} \right]^{-1} \tag{12.18}$$

The probability of delaying (no channel is available) is

$$\text{prob[queueing]} \triangleq C(N,A) = \sum_{k=N}^{\infty} p_k$$

Using p_k for $k \geq N$ we obtain

$$C(N,A) = \frac{A^N}{N!} \frac{1}{1 - A/N} p_0 \tag{12.19}$$

The above equation is known as Erlang-C formula.

Finite Population

M customers

$$\lambda_k = (M - k)\lambda \quad , \qquad k \leq N - 1$$

$$\mu_k = k\mu \qquad , \qquad k < N$$

Accordingly,

$$p_k = \frac{\binom{M}{k} A^k}{\sum_{i=0}^{N} \binom{M}{i} A^i} \tag{12.20}$$

This is known as the Engset distribution.

APPENDIX 12C

TABLE 12C.1. Blocked Calls Cleared — Erlang B

	A (erlangs)													
	Blocking Probability													
N	1.0%	1.2%	1.5%	2%	3%	5%	7%	10%	15%	20%	30%	40%	50%	
1	0.0101	0.0121	0.0152	0.0204	0.0309	0.0526	0.0753	0.111	0.176	0.250	0.429	0.667	1.00	
2	0.153	0.168	0.190	0.223	0.282	0.381	0.470	0.595	0.796	1.00	1.45	2.00	2.73	
3	0.455	0.489	0.535	0.602	0.715	0.899	1.06	1.27	1.60	1.93	2.63	3.48	4.59	
4	0.869	0.922	0.992	1.09	1.26	1.52	1.75	2.05	2.50	2.95	3.89	5.02	6.50	
5	1.36	1.43	1.52	1.66	1.88	2.22	2.50	2.88	3.45	4.01	5.19	6.60	8.44	
6	1.91	2.00	2.11	2.28	2.54	2.96	3.30	3.76	4.44	5.11	6.51	8.19	10.4	
7	2.50	2.60	2.74	2.94	3.25	3.74	4.14	4.67	5.46	6.23	7.86	9.80	12.4	
8	3.13	3.25	3.40	3.63	3.99	4.54	5.00	5.60	6.50	7.37	9.21	11.4	14.3	
9	3.78	3.92	4.09	4.34	4.75	5.37	5.88	6.55	7.55	8.52	10.6	13.0	16.3	
10	4.46	4.61	4.81	5.08	5.53	6.22	6.78	7.51	8.62	9.68	12.0	14.7	18.3	
11	5.16	5.32	5.54	5.84	6.33	7.08	7.69	8.49	9.69	10.9	13.3	16.3	20.3	
12	5.88	6.05	6.29	6.61	7.14	7.95	8.61	9.47	10.8	12.0	14.7	18.0	22.2	
13	6.61	6.80	7.05	7.40	7.97	8.83	9.54	10.5	11.9	13.2	16.1	19.6	24.2	
14	7.35	7.56	7.82	8.20	8.80	9.73	10.5	11.5	13.0	14.4	17.5	21.2	26.2	
15	8.11	8.33	8.61	9.01	9.65	10.6	11.4	12.5	14.1	15.6	18.9	22.9	28.2	
16	8.88	9.11	9.41	9.83	10.5	11.5	12.4	13.5	15.2	16.8	20.3	24.5	30.2	
17	9.65	9.89	10.2	10.7	11.4	12.5	13.4	14.5	16.3	18.0	21.7	26.2	32.2	
18	10.4	10.7	11.0	11.5	12.2	13.4	14.3	15.5	17.4	19.2	23.1	27.8	34.2	
19	11.2	11.5	11.8	12.3	13.1	14.3	15.3	16.6	18.5	20.4	24.5	29.5	36.2	
20	12.0	12.3	12.7	13.2	14.0	15.2	16.3	17.6	19.6	21.6	25.9	31.2	38.2	
21	12.8	13.1	13.5	14.0	14.9	16.2	17.3	18.7	20.8	22.8	27.3	32.8	40.2	
22	13.7	14.0	14.3	14.9	15.8	17.1	18.2	19.7	21.9	24.1	28.7	34.5	42.1	
23	14.5	14.8	15.2	15.8	16.7	18.1	19.2	20.7	23.0	25.3	30.1	36.1	44.1	
24	15.3	15.6	16.0	16.6	17.6	19.0	20.2	21.8	24.2	26.5	31.6	37.8	46.1	
25	16.1	16.5	16.9	17.5	18.5	20.0	21.2	22.8	25.3	27.7	33.0	39.4	48.1	
26	17.0	17.3	17.8	18.4	19.4	20.9	22.2	23.9	26.4	28.9	34.4	41.1	50.1	
27	17.8	18.2	18.6	19.3	20.3	21.9	23.2	24.9	27.6	30.2	35.8	42.8	52.1	
28	18.6	19.0	19.5	20.2	21.2	22.9	~~25.2~~	26.0	28.7	31.4	37.2	44.4	54.1	24.2
29	19.5	19.9	20.4	21.0	22.1	23.8	25.2	27.1	29.9	32.6	38.6	46.1	56.1	
30	20.3	20.7	21.2	21.9	23.1	24.8	26.2	28.1	31.0	33.8	40.0	47.7	58.1	
31	21.2	21.6	22.1	22.8	24.0	25.8	27.2	29.2	32.1	35.1	41.5	49.4	60.1	
32	22.0	22.5	23.0	23.7	24.9	26.7	28.2	30.2	33.3	36.3	42.9	51.1	62.1	
33	22.9	23.3	23.9	24.6	25.8	27.7	29.3	31.3	34.4	37.5	44.3	52.7	64.1	
34	23.8	24.2	24.8	25.5	26.8	28.7	30.3	32.4	35.6	38.8	45.7	54.4	66.1	
35	24.6	25.1	25.6	26.4	27.7	29.7	31.3	33.4	36.7	40.0	47.1	56.0	68.1	
36	25.5	26.0	26.5	27.3	28.6	30.7	~~32.2~~	34.5	37.9	41.2	48.6	57.7	70.1	32.3
37	26.4	26.8	27.4	28.3	29.6	31.6	33.3	35.6	39.0	42.4	50.0	59.4	72.1	
38	27.3	27.7	28.3	29.2	30.5	32.6	34.4	36.6	40.2	43.7	51.4	61.0	74.1	
39	28.1	28.6	29.2	30.1	31.5	33.6	35.4	37.7	41.3	44.9	52.8	62.7	76.1	
40	29.0	29.5	30.1	31.0	32.4	~~35.6~~	36.4	38.8	42.5	46.1	54.2	64.4	78.1	34.6
41	29.9	30.4	31.0	31.9	33.4	35.6	37.4	39.9	43.6	47.4	55.7	66.0	80.1	
42	30.8	31.3	31.9	32.8	34.3	36.6	38.4	40.9	44.8	48.6	57.1	67.7	82.1	
43	31.7	32.2	32.8	33.8	35.3	37.6	39.5	42.0	45.9	49.9	58.5	69.3	84.1	
44	32.5	33.1	33.7	34.7	36.2	38.6	40.5	43.1	47.1	51.1	59.9	71.0	86.1	
45	33.4	34.0	34.6	35.6	37.2	39.6	41.5	44.2	48.2	52.3	61.3	72.7	88.1	
46	34.3	34.9	35.6	36.5	38.1	40.5	42.6	45.2	49.4	53.6	62.8	74.3	90.1	
47	35.2	35.8	36.5	37.5	39.1	41.5	43.6	46.3	50.6	54.8	64.2	76.0	92.1	
48	36.1	36.7	37.4	38.4	40.0	42.5	44.6	47.4	51.7	56.0	65.6	77.7	94.1	
49	37.0	37.6	38.3	39.3	41.0	43.5	45.7	48.5	52.9	57.3	67.0	79.3	96.1	
50	37.9	38.5	39.2	40.3	41.9	44.5	46.7	49.6	54.0	58.5	68.5	81.0	98.1	
51	38.8	39.4	40.1	41.2	42.9	45.5	47.7	50.6	55.2	59.7	69.9	82.7	100.1	
52	39.7	40.3	41.0	42.1	43.9	46.5	48.8	51.7	56.3	61.0	71.3	84.3	102.1	
53	40.6	41.2	42.0	43.1	44.8	47.5	49.8	52.8	57.5	62.2	72.7	86.0	104.1	
54	41.5	42.1	42.9	44.0	45.8	48.5	50.8	53.9	58.7	63.5	74.2	87.6	106.1	
55	42.4	43.0	43.8	44.9	46.7	49.5	51.9	55.0	59.8	64.7	75.6	89.3	108.1	
56	43.3	43.9	44.7	45.9	47.7	50.5	52.9	56.1	61.0	65.9	77.0	91.0	110.1	
57	44.2	44.8	45.7	46.8	48.7	51.5	53.9	57.1	62.1	67.2	78.4	92.6	112.1	
58	45.1	45.8	46.6	47.8	49.6	52.6	55.0	58.2	63.3	68.4	79.8	94.3	114.1	

TABLE 12C.1.

	A (erlangs)													
	Blocking Probability													
N	1.0%	1.2%	1.5%	2%	3%	5%	7%	10%	15%	20%	30%	40%	50%	
58	45.1	45.8	46.6	47.8	49.6	52.6	55.0	58.2	63.3	68.4	79.8	94.3	114.1	
59	46.0	46.7	47.5	48.7	50.6	53.6	~~45.0~~	59.3	64.5	69.7	81.3	96.0	116.1	56.0
60	46.9	47.6	48.4	49.6	51.6	54.6	57.1	60.4	65.6	70.9	82.7	97.6	118.1	
61	47.9	48.5	49.4	50.6	52.5	55.6	58.1	61.5	66.8	72.1	84.1	99.3	120.1	
62	48.8	49.4	50.3	51.5	53.5	56.6	59.1	62.6	68.0	73.4	85.5	101.0	122.1	
63	49.7	50.4	51.2	52.5	54.5	57.6	60.2	63.7	69.1	74.6	87.0	102.6	124.1	
64	50.6	51.3	52.2	53.4	55.4	58.6	61.2	64.8	70.3	75.9	88.4	104.3	126.1	
65	51.5	52.2	53.1	54.4	56.4	59.6	62.3	65.8	71.4	77.1	89.8	106.0	128.1	
66	52.4	53.1	54.0	55.3	57.4	60.6	63.3	66.9	72.6	78.3	91.2	107.6	130.1	
67	53.4	54.1	55.0	56.3	58.4	61.6	64.4	68.0	73.8	79.6	92.7	109.3	132.1	
68	54.3	55.0	55.9	57.2	59.3	62.6	65.4	69.1	74.9	80.8	94.1	111.0	134.1	
69	55.2	55.9	56.9	58.2	60.3	63.7	66.4	70.2	76.1	82.1	95.5	112.6	136.1	
70	56.1	56.8	57.8	59.1	61.3	64.7	67.5	71.3	77.3	83.3	96.9	114.3	138.1	
71	57.0	57.8	58.7	60.1	62.3	65.7	68.5	72.4	78.4	84.6	98.4	115.9	140.1	
72	58.0	58.7	59.7	61.0	63.2	66.7	69.6	73.5	79.6	85.8	99.8	117.6	142.1	
73	58.9	59.6	60.6	62.0	64.2	67.7	70.6	74.6	80.8	87.0	101.2	119.3	144.1	
74	59.8	60.6	61.6	62.9	65.2	68.7	71.7	75.6	81.9	88.3	102.7	120.9	146.1	
75	60.7	61.5	62.5	63.9	66.2	69.7	72.7	76.7	83.1	89.5	104.1	122.6	148.0	
76	61.7	62.4	63.4	64.9	67.2	70.8	73.8	77.8	84.2	90.8	105.5	124.3	150.0	
77	62.6	63.4	64.4	65.8	68.1	71.8	74.8	78.9	85.4	92.0	106.9	~~[illegible]~~	152.0	125.9
78	63.5	64.3	65.3	66.8	69.1	72.8	75.9	80.0	86.6	93.3	108.4	127.6	154.0	
79	64.4	65.2	66.3	67.7	70.1	73.8	76.9	81.1	87.7	94.5	109.8	129.3	156.0	
80	65.4	66.2	67.2	68.7	71.1	74.8	78.0	82.2	88.9	95.7	111.2	130.9	158.0	
81	66.3	67.1	68.2	69.6	72.1	75.8	79.0	83.3	90.1	97.0	112.6	132.6	160.0	
82	67.2	68.0	69.1	70.6	73.0	76.9	80.1	84.4	91.2	98.2	114.1	134.3	162.0	
83	68.2	69.0	70.1	71.6	74.0	77.9	81.1	85.5	92.4	99.5	115.5	135.9	164.0	
84	69.1	69.9	71.0	72.5	75.0	78.9	82.2	86.6	93.6	100.7	116.9	137.6	166.0	
85	70.0	70.9	71.9	73.5	76.0	79.9	83.2	87.7	94.7	102.0	118.3	139.3	168.0	
86	70.9	71.8	72.9	74.5	77.0	80.9	84.3	88.8	95.9	103.2	119.8	140.9	170.0	
87	71.9	72.7	73.8	75.4	78.0	82.0	85.3	89.9	97.1	104.5	121.2	142.6	172.0	
88	72.8	73.7	74.8	76.4	78.9	83.0	86.4	91.0	98.2	105.7	122.6	144.6	174.0	
89	73.7	74.6	75.7	77.3	79.9	84.0	87.4	92.1	99.4	106.9	124.0	145.9	176.0	
90	74.7	75.6	76.7	78.3	80.9	85.0	88.5	93.1	100.6	108.2	125.5	147.6	178.0	
91	75.6	76.5	77.6	79.3	81.9	86.0	89.5	94.2	101.7	109.4	126.9	149.3	180.0	
92	76.6	77.4	78.6	80.2	82.9	87.1	90.6	95.3	102.9	110.7	128.3	150.9	182.0	
93	77.5	78.4	79.6	81.2	83.9	88.1	91.6	96.4	104.1	111.9	129.7	152.6	184.0	
94	78.4	79.3	80.5	82.2	84.9	89.1	92.7	97.5	105.3	113.2	131.2	154.3	186.0	
95	79.4	80.3	81.5	83.1	85.8	90.1	93.7	98.6	106.4	114.4	132.6	155.9	188.0	
96	80.3	81.2	82.4	84.1	86.8	91.1	94.8	99.7	107.6	115.7	134.0	157.6	190.0	
97	81.2	82.2	83.4	85.1	87.8	92.2	95.8	100.8	108.8	116.9	135.5	159.3	192.0	
98	82.2	83.1	84.3	86.0	88.8	93.2	96.9	101.9	109.9	118.2	136.9	160.9	194.0	
99	83.1	84.1	85.3	87.0	89.8	94.2	97.9	103.0	111.1	119.4	138.3	162.6	196.0	
100	84.1	85.0	86.2	88.0	90.8	95.2	99.0	104.1	112.3	120.6	139.7	164.3	198.0	
102	85.9	86.9	88.1	89.9	92.8	97.3	101.1	106.3	114.6	123.1	142.6	167.6	202.0	
104	87.8	88.8	90.1	91.9	94.8	99.3	103.2	108.5	116.9	125.6	145.4	170.9	206.0	
106	89.7	90.7	92.0	93.8	96.7	101.4	105.3	110.7	119.3	128.1	148.3	174.2	210.0	
108	91.6	92.6	93.9	95.7	98.7	103.4	107.4	112.9	121.6	130.6	151.1	177.6	214.0	
110	93.5	94.5	95.8	97.7	100.7	105.5	109.5	115.1	124.0	133.1	154.0	180.9	218.0	
112	95.4	96.4	97.7	99.6	102.7	107.5	111.7	117.3	126.3	135.6	156.9	184.2	222.0	
114	97.3	98.3	99.7	101.6	104.7	109.6	113.8	119.5	128.6	138.1	159.7	187.6	226.0	
116	99.2	100.2	101.6	103.5	106.7	111.7	115.9	121.7	131.0	140.6	162.6	190.9	230.0	
118	101.1	102.1	103.5	105.5	108.7	113.7	118.0	123.9	133.3	143.1	165.4	194.2	234.0	
120	103.0	104.0	105.4	107.4	110.7	115.8	120.1	126.1	135.7	145.6	168.3	197.6	238.0	
122	104.9	105.9	107.4	109.4	112.6	117.8	122.2	128.3	138.0	148.1	171.1	200.9	242.0	
124	106.8	107.9	109.3	111.3	114.6	119.9	124.4	130.5	140.3	150.6	174.0	204.2	246.0	
126	108.7	109.8	111.2	113.3	116.6	121.9	126.5	132.7	142.7	153.0	176.8	207.6	250.0	
128	110.6	111.7	113.2	115.2	118.6	124.0	128.6	134.9	145.0	155.5	179.7	210.9	254.0	
130	112.5	113.6	115.1	117.2	120.6	126.1	130.7	137.1	147.4	158.0	182.5	214.2	258.0	
132	114.4	115.5	117.0	119.1	122.6	128.1	132.8	139.3	149.7	160.5	185.4	217.6	262.0	
134	116.3	117.4	119.0	121.1	124.6	130.2	134.9	141.5	152.0	163.0	188.3	220.9	266.0	
136	118.2	119.4	120.9	123.1	126.6	132.3	137.1	143.7	154.4	165.5	191.1	224.2	270.0	

TABLE 12C.1.

	A (erlangs)													
	Blocking Probability													
N	1.0%	1.2%	1.5%	2%	3%	5%	7%	10%	15%	20%	30%	40%	50%	
138	120.1	121.3	122.8	125.0	128.6	134.3	139.2	145.9	156.7	168.0	194.0	227.6	274.0	
140	122.0	123.2	124.8	127.0	130.6	136.4	141.3	148.1	159.1	170.5	196.8	230.9	278.0	
142	123.9	125.1	126.7	128.9	132.6	138.4	143.4	150.3	161.4	173.0	199.7	234.2	282.0	
144	125.8	127.0	128.6	130.9	134.6	140.5	145.6	152.5	163.8	175.5	202.5	237.6	286.0	
146	127.7	129.0	130.6	132.9	136.6	142.6	147.7	154.7	166.1	178.0	205.4	240.9	290.0	
148	129.7	130.9	132.5	134.8	138.6	144.6	149.8	156.9	168.5	180.5	208.2	244.2	294.0	
150	131.6	132.8	134.5	136.8	140.6	146.7	151.9	159.1	170.8	183.0	211.1	247.6	298.0	
152	133.5	134.8	136.4	138.8	142.6	148.8	154.0	161.3	173.1	185.5	214.0	250.9	302.0	
154	135.4	136.7	138.4	140.7	144.6	150.8	156.2	163.5	175.5	188.0	216.8	254.2	306.0	
156	137.3	138.6	140.3	142.7	146.6	152.9	158.3	165.7	177.8	190.5	219.7	257.6	310.0	
158	139.2	140.5	142.3	144.7	148.6	155.0	160.4	167.9	180.2	193.0	222.5	260.9	314.0	
160	141.2	142.5	144.2	146.6	150.6	157.0	162.5	170.2	182.5	195.5	225.4	264.2	318.0	
162	143.1	144.4	146.1	148.6	152.7	159.1	164.7	172.4	184.9	198.0	228.2	267.6	322.0	
164	145.0	146.3	148.1	150.6	154.7	161.2	166.8	174.6	187.2	200.4	231.1	270.9	326.0	
166	146.9	148.3	150.0	152.6	156.7	163.3	168.9	176.8	189.6	202.9	233.9	274.2	330.0	
168	148.9	150.2	152.0	154.5	158.7	165.3	171.0	179.0	191.9	205.4	236.8	277.6	334.0	
170	150.8	152.1	153.9	156.5	160.7	167.4	173.2	181.2	194.2	207.9	239.7	280.9	338.0	
172	152.7	154.1	155.9	158.5	162.7	169.5	175.3	183.4	196.6	210.4	242.5	284.2	342.0	
174	154.6	156.0	157.8	160.4	164.7	171.5	177.4	185.6	198.9	212.9	245.4	287.6	346.0	
176	156.6	158.0	159.8	162.4	166.7	173.6	179.6	187.8	201.3	215.4	248.2	290.9	350.0	
178	158.5	159.9	161.8	164.4	168.7	175.7	181.7	190.0	203.6	217.9	251.1	294.2	354.0	
180	160.4	161.8	163.7	166.4	170.7	177.8	183.8	192.2	206.0	220.4	253.9	297.5	358.0	
182	162.3	163.8	165.7	168.3	172.8	179.8	185.9	194.4	208.3	222.9	256.8	300.9	362.0	
184	164.3	165.7	167.6	170.3	174.8	181.9	188.1	196.6	210.7	225.4	259.6	304.2	366.0	
186	166.2	167.7	169.6	172.3	176.8	184.0	190.2	198.9	213.0	227.9	262.5	307.5	370.0	
188	168.1	169.6	171.5	174.3	178.8	186.1	192.3	201.1	215.4	230.4	265.4	310.9	374.0	
190	170.1	171.5	173.5	176.3	180.8	188.1	194.5	203.3	217.7	232.9	268.2	314.2	378.0	
192	172.0	173.5	175.4	178.2	182.8	190.2	196.6	205.5	220.1	235.4	271.1	317.5	382.0	
194	173.9	175.4	177.4	180.2	184.8	192.3	198.7	207.7	222.4	237.9	273.9	320.9	386.0	
196	175.9	177.4	179.4	182.2	186.9	194.4	200.8	209.9	224.8	240.4	276.8	324.2	390.0	
198	177.8	179.3	181.3	184.2	188.9	196.4	203.0	212.1	227.1	242.9	279.6	327.5	394.0	
200	179.7	181.3	183.3	186.2	190.9	198.5	205.1	214.3	229.4	245.4	282.5	330.9	398.0	
202	181.7	183.2	185.2	188.1	192.9	200.6	207.2	216.5	231.8	247.9	285.4	334.2	402.0	
204	183.6	185.2	187.2	190.1	194.9	202.7	209.4	218.7	234.1	250.4	288.2	337.5	~~306.0~~	406.0
206	185.5	187.1	189.2	192.1	196.9	204.7	211.5	221.0	236.5	252.9	291.1	340.9	~~310.0~~	410.0
208	187.5	189.1	191.1	194.1	199.0	206.8	213.6	223.2	238.8	255.4	293.9	344.2	414.0	
210	189.4	191.0	193.1	196.1	201.0	208.9	215.8	225.4	241.2	257.9	296.8	347.5	418.0	
212	191.4	193.0	195.1	198.1	203.0	211.0	217.9	227.6	243.5	260.4	299.6	350.9	422.0	
214	193.3	194.9	197.0	200.0	205.0	213.0	220.0	229.8	245.9	262.9	302.5	354.2	426.0	
216	195.2	196.9	199.0	202.0	207.0	215.1	222.2	232.0	248.2	265.4	305.3	357.5	430.0	
218	197.2	198.8	201.0	204.0	209.1	217.2	224.3	234.2	250.6	267.9	308.2	360.9	434.0	
220	199.1	200.8	202.9	206.0	211.1	219.3	226.4	236.4	252.9	270.4	311.1	364.2	438.0	
222	201.1	202.7	204.9	208.0	213.1	221.4	228.6	238.6	255.3	272.9	313.9	367.5	442.0	
224	203.0	204.7	206.8	210.0	215.1	223.4	230.7	240.9	257.6	275.4	316.8	370.9	446.0	
226	204.9	206.6	208.8	212.0	217.1	225.5	232.8	243.1	260.0	277.8	319.6	374.2	450.0	
228	206.9	208.6	210.8	213.9	219.2	227.6	235.0	245.3	262.3	280.3	322.5	377.5	454.0	
230	208.8	210.5	212.8	215.9	221.2	229.7	237.1	247.5	264.7	282.8	325.3	380.9	458.0	
232	210.8	212.5	214.7	217.9	223.2	231.8	239.2	249.7	267.0	285.3	328.2	384.2	462.0	
234	212.7	214.4	216.7	219.9	225.2	233.8	241.4	251.9	269.4	287.8	331.1	387.5	466.0	
236	214.7	216.4	218.7	221.9	227.2	235.9	243.5	254.1	271.7	290.3	333.9	390.9	470.0	
238	216.6	218.3	220.6	223.9	229.3	238.0	245.6	256.3	274.1	292.8	336.8	394.2	474.0	
240	218.6	220.3	222.6	225.9	231.3	240.1	247.8	258.6	276.4	295.3	339.6	397.5	478.0	
242	220.5	222.3	224.6	227.9	233.3	242.2	249.9	260.8	278.8	297.8	342.5	400.9	482.0	
244	222.5	224.2	226.5	229.9	235.3	244.3	252.0	263.0	281.1	300.3	345.3	404.2	~~386.0~~	486.0
246	224.4	226.2	228.5	231.8	237.4	246.3	254.2	265.2	283.4	~~402.8~~	348.2	407.5	490.0	302.8
248	226.3	228.1	230.5	233.8	239.4	248.4	256.3	267.4	285.8	305.3	351.0	410.9	494.0	
250	228.3	230.1	232.5	235.8	241.4	250.5	258.4	269.6	288.1	307.8	353.9	414.2	498.0	
	0.976	0.982	0.988	0.988	1.014	1.042	1.070	1.108	1.176	1.250	1.428	1.666	2.000	
300	277.1	279.2	281.9	285.7	292.1	302.6	311.9	325.0	346.9	370.3	425.3	497.5	598.0	
	0.982	0.984	0.990	1.000	1.016	1.044	1.070	1.108	1.174	1.248	1.428	1.668	2.000	

TABLE 12C.1.

	A (erlangs)													
	Blocking Probability													
N	1.0%	1.2%	1.5%	2%	3%	5%	7%	10%	15%	20%	30%	40%	50%	
350	326.2	328.4	331.4	335.7	342.9	354.8	365.4	380.4	405.6	432.7	496.7	580.9	~~598.0~~	698.0
	0.982	0.988	0.994	1.004	1.020	1.046	1.070	1.108	1.176	1.250	1.430	1.666	2.000	
400	375.3	377.8	381.1	385.9	393.9	407.1	418.9	435.8	464.4	495.2	568.2	664.2	~~698.0~~	798.0
	0.986	0.990	0.996	1.004	1.018	1.046	1.072	1.110	1.176	1.250	1.428	1.666	2.000	
450	424.6	427.3	430.9	~~336.1~~	444.8	459.4	472.5	491.3	523.2	557.7	639.6	747.5	898.0	436.1
	0.988	0.994	0.998	1.006	1.022	1.048	1.070	1.108	1.176	1.250	1.428	1.668	2.000	
500	474.0	477.0	480.8	486.4	495.9	511.8	526.0	546.7	582.0	620.2	711.0	830.9	998.0	
	0.991	0.994	1.000	1.008	1.022	1.047	1.073	1.110	1.176	1.249	1.429	1.666	2.000	
600	573.1	576.4	580.8	587.2	598.1	616.5	633.3	657.7	699.6	745.1	853.9	997.5	1198.	
	0.993	0.997	1.002	1.010	1.024	1.049	1.073	1.110	1.176	1.250	1.428	1.665	2.00	
700	672.4	676.1	681.0	688.2	700.5	721.4	740.6	768.7	817.2	870.1	996.7	1164.	1398.	
	0.994	0.998	1.004	1.011	1.025	1.050	1.073	1.110	1.176	1.250	1.433	1.67	2.00	
800	771.8	775.9	781.4	789.3	803.0	826.4	847.9	879.7	934.8	995.1	1140.	1331.	1598.	
	0.997	1.000	1.004	1.013	1.025	1.050	1.074	1.111	1.172	1.249	1.42	1.67	2.00	
900	871.5	875.9	881.8	890.6	905.5	931.4	955.3	990.8	1052.	1120.	1282.	1498.	1798.	
	0.997	1.001	1.006	1.013	1.025	1.046	1.077	1.112	1.18	1.25	1.43	1.66	2.00	
1000	971.2	976.0	982.4	991.9	1008.	1036.	1063.	1102.	1170.	1245.	1425.	1664.	1998.	
	0.998	1.000	1.006	1.011	1.03	1.05	1.07	1.11	1.18	1.25	1.43	1.67	2.00	
1100	1071.	1076.	1083.	1093.	1111.	1141.	1170.	1213.	1288.	1370.	1568.	1831.	2198.	(missing dots)

Source: From *Telephone Traffic Theory Tables and Charts*, Siemens Aktiengesellschaft, Munich, 1970.

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