SINR Bounds for Broadcast Channels with Zero-Forcing Beamforming and Limited Feedback

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***Abstract*—Channel state information at the transmitter (CSIT) enables high rates in multiuser systems with multiple transmit antennas, as it can be used for multiplexing gain and multiuser diversity. However, CSIT is often obtained through information fedback from the users, and it can be severely quantized to limit overhead. In this case, transmitting to multiple users will typically cause interference among them, which degrades the signal-to-interference-plus-noise ratio (SINR). In this letter, we derive bounds on this SINR, assuming that the transmitter uses zero-forcing beamforming, and that active users are almost orthogonal to one another. Simulation results show that our bounds may be used as a metric for user scheduling.**

***Index Terms*—Broadcast channels, array signal processing, feedback communication, cochannel interference, diversity methods.**

I. Introduction

CHANNEL state information at the transmitter (CSIT) is crucial for high-rate transmission in wireless systems with multiple transmit antennas [1]. Indeed, CSIT can be used for interference mitigation, allowing the system to serve several users simultaneously. Also, CSIT allows the system to transmit to the users with the most favorable channels, achieving multiuser diversity. However, CSIT is not easily available in practice. For instance, in frequency-division duplex systems, the users must estimate the channel and feed it back to the base station (BS). To limit the overhead incurred by this feedback, severe quantization may be employed. In this case, the system must be able to exploit a very imprecise CSIT.

In [2], the authors propose a method to exploit limited CSIT in multiple-input-single-output (MISO) broadcast channels. The feedback is divided into channel quality information (CQI) and channel direction information (CDI). The CDI, corresponding to the quantization of the normalized channel vector of the users, is used by the BS to perform zero-forcing beamforming (ZFBF), enabling simultaneous transmission to several users. The CQI, which is related to the signal-to-interference-plus-noise ratio (SINR), is used for user scheduling and rate allocation. However, the actual SINR cannot be computed at the user terminals, as it depends on the beamforming vectors, which are known only at the BS after scheduling. To circumvent this problem, [2] proposes a CQI metric based on a bound on the expected value of the interference of each link.

In this paper we derive two lower bounds on the SINR resulting from the use of ZFBF with limited feedback. In contrast with [2], we bound the actual SINR, not its expected value. One of the bounds is tight, and is based on information available at the users. The other bound improves on the first by using information available at the transmitter after user selection. Although we could not prove that this second bound is tight, we can prove that, by using the knowledge of which users were scheduled, it always improves on the first bound. As in [2], we assume that users are served with equal power and that the scheduling algorithm is constrained to selecting users that are all "almost" orthogonal. (This concept will be made precise in the sequel.) Besides [2], our derivations also share some similarities with [3] and [4]. However, the bounds in [3] and [4] are used mostly for user selection, and do not provide good results when used for rate adaptation. This paper extends [5] by the same authors, deriving tighter and more general lower bounds.

This paper is organized as follows: In Section II, we describe our system model and the problem to be treated. In Section III we derive the SINR lower bounds. Section IV shows some numerical results, and Section V is the conclusion.

II. System Model and Problem Statement

The channel model used in this paper is the wireless MISO broadcast channel[1], composed of one BS with an array of N_t transmit antennas, and K user terminals, each with one antenna. We assume block, flat fading. At each block the BS transmits to $M \leq N_t$ equal-power users, and performs ZFBF to separate them spatially. Both beamforming and scheduling are based on information fed back by the users through a delay- and error-free channel. We assume perfect knowledge of the channel at the receivers.

In each block the k-th scheduled user receives the signal

$$y_k = \sqrt{\rho}\, \mathbf{w}_k^{\mathrm{H}}\, \mathbf{h}_k\, x_k + \sqrt{\rho} \sum_{j=1, j\neq k}^{M} \mathbf{w}_j^{\mathrm{H}}\, \mathbf{h}_k\, x_j + n_k, \qquad (1)$$

where $\mathbf{h}_k$ is the $N_t \times 1$ channel vector of the k-th user, $\mathbf{w}_k$ is the beamforming vector, x_k is the signal intended for the k-th user, and n_k is additive white noise with distribution $\mathcal{N}(0, \sigma^2)$. We assume that $\mathrm{E}[|x_k|^2] = 1$ and $\|\mathbf{w}_k\| = 1$, so that ρ is the power allocated to each scheduled user. The power constraint

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[1]The bounds proposed here could be easily extended to MIMO channel using, for instance, the method proposed in [6].

may be written as $M\rho \leq P$, where P is the total transmit power.

As in [2], we divide channel state feedback into channel quality and channel direction information. The CQI is normally given in terms of the SINR [2], which, from (1), is given by

$$\gamma_k = \frac{\rho\,|\mathbf{w}_k^{\mathrm{H}}\,\mathbf{h}_k|^2}{\sigma^2 + \rho\,\sum_{j=1,j\neq k}^{M} |\mathbf{w}_j^{\mathrm{H}}\,\mathbf{h}_k|^2}. \tag{2}$$

The CDI, on the other hand, is used at the BS for zero-forcing beamforming. When full CSIT is available, the unit-norm beamforming vector of the k-th user is orthogonal to the channels of all other scheduled users[2]. In other words, $|\mathbf{w}_k^{\mathrm{H}}\mathbf{h}_j| = 0$ when $j \neq k$. Note that only the channel direction is relevant for this orthogonality condition, so this is the quantity that is quantized and fedback. The quantization of CDI is performed by picking the vector out of a unit-norm codebook $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \ldots \mathbf{c}_N\}$[3] that shows the best alignment with the channel [2]. Mathematically, the quantized channel of the k-th user, $\hat{\mathbf{h}}_k$, is given by $\hat{\mathbf{h}}_k = \arg\max_{\mathbf{c}_i} |\mathbf{h}_k^{\mathrm{H}}\mathbf{c}_i|$.

Finally, we assume that the scheduling algorithm only schedules users whose quantized channels are ϵ-orthogonal [7]. In other words, the quantized channels of scheduled users k and j satisfy

$$|\hat{\mathbf{h}}_k^{\mathrm{H}}\hat{\mathbf{h}}_j| \leq \epsilon_h \qquad \forall \quad j \neq k. \tag{3}$$

We assume that M and ϵ_h are system parameters known by all users.

As can be seen from (2), the SINR cannot be computed at any terminal. The BS has no access to $\mathbf{h}_k$, whereas the users have no access to $\mathbf{w}_k$, which are calculated at the BS after user scheduling. In the next section, we determine a lower bound on the SINR that may be computed at the user terminal. Then, we propose a correction factor, applied by the BS after determining the beamforming vectors, that improves the bound.

III. Lower Bounds on the SINR

In this section, we derive lower bounds on the SINR in (2) under the system model described in Section II. We begin by rewriting (2). To that end, we decompose the channel vector of the k-th user as

$$\mathbf{h}_k = ||\mathbf{h}_k||\,\tilde{\mathbf{h}}_k = ||\mathbf{h}_k||\,(a_k\,\hat{\mathbf{h}}_k + \bar{a}_k\mathbf{e}_k), \tag{4}$$

where $\tilde{\mathbf{h}}_k$ is the normalized channel vector, a_k is the projection of $\tilde{\mathbf{h}}_k$ onto the quantized CDI $\hat{\mathbf{h}}_k$, $\mathbf{e}_k$ is a unit vector orthogonal to $\hat{\mathbf{h}}_k$, and $\bar{a}_k$ is the projection of $\tilde{\mathbf{h}}_k$ onto $\mathbf{e}_k$. Also, note that the ZFBF vectors are chosen by the BS based on the quantized CDI. Thus, instead of satisfying $|\mathbf{w}_k^{\mathrm{H}}\mathbf{h}_j| = 0$, the ZFBF vectors satisfy

$$|\mathbf{w}_j^{\mathrm{H}}\hat{\mathbf{h}}_k| = 0 \qquad \forall \quad j \neq k, \tag{5}$$

which leads to residual interference between the users. Using (4) and (5), the instantaneous SINR with the beamforming vectors in (5) can be written as

$$\gamma_k = \frac{\rho\,\|\mathbf{h}_k\|^2\,|a_k\,\mathbf{w}_k^{\mathrm{H}}\,\hat{\mathbf{h}}_k + \bar{a}_k\,\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|^2}{\sigma^2 + \rho\,\|\mathbf{h}_k\|^2\,\sum_{j=1,j\neq k}^{M} |\bar{a}_k\,\mathbf{w}_j^{\mathrm{H}}\,\mathbf{e}_k|^2}. \tag{6}$$

We now derive bounds on (6).

A. Tight Lower Bound at the User Terminal

In this section, we derive a tight lower bound on the SINR, based on the information available at the user terminals. To prove that the bound is tight, we will construct codebook vectors and a channel realization that result in an SINR equal to the bound. Our proof is based on the following lemmas.

Lemma 1: Assume that the quantized channels are ϵ-orthogonal, and that the ZFBF vectors $\mathbf{w}_k$ satisfy (5). Then, $|\mathbf{w}_i^{\mathrm{H}}\,\mathbf{w}_j| \leq \min(\epsilon_w, 1)$ for $i \neq j$, where

$$\epsilon_w = \frac{\epsilon_h}{1 - (M-2)\,\epsilon_h}. \tag{7}$$

Proof: See Appendix A, where we also show that the bound is tight. ■

Lemma 2: If the beamforming vectors satisfy $|\mathbf{w}_i^{\mathrm{H}}\mathbf{w}_k| \leq \epsilon_w$, with ϵ_w given by (7), then the residual interference term in (6) is upper-bounded by

$$\sum_{j=1,j\neq k}^{M} |\mathbf{w}_j^{\mathrm{H}}\,\mathbf{e}_k|^2 \leq 1 + (M-2)\,\epsilon_w. \tag{8}$$

Proof: See Appendix B, where we also show that the bound is tight. ■

Lemma 3: If the quantized channel vectors are ϵ-orthogonal, then

$$|\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|^2 \leq \bar{B}^2 \text{ and } |\mathbf{w}_k^{\mathrm{H}}\,\hat{\mathbf{h}}_k|^2 \geq B^2, \tag{9}$$

where $\bar{B}^2 \triangleq 1 - B^2$ and

$$B^2 \triangleq (1+\epsilon_h)\frac{1-(M-1)\,\epsilon_h}{1-(M-2)\,\epsilon_h}. \tag{10}$$

Proof: The value of B already appears without proof in [2]. In appendix C, we provide a simple derivation of this bound. We also show that it is tight. ■

We are now ready to derive a tight lower bound on the SINR of the link between the BS and the k-th user.

Theorem 1: Let $\alpha_k = |a_k| = |\tilde{\mathbf{h}}_k^{\mathrm{H}}\,\hat{\mathbf{h}}_k|$, and $\bar{\alpha}_k = |\bar{a}_k| = |\tilde{\mathbf{h}}_k^{\mathrm{H}}\,\mathbf{e}_k|$. Assume that M users are scheduled to transmit with equal power using ZFBF. Then, if $|\hat{\mathbf{h}}_k^{\mathrm{H}}\hat{\mathbf{h}}_j| \leq \epsilon_h$ for all of the M scheduled users, the SINR γ_k is lower bounded by γ_{LB_k}, i.e., $\gamma_k \geq \gamma_{LB_k}$, where

$$\gamma_{LB_k} = \frac{\frac{P}{M}\,\|\mathbf{h}_k\|^2\,(\alpha_k\,B - \bar{\alpha}_k\bar{B})^2}{\sigma^2 + \frac{P}{M}\,\|\mathbf{h}_k\|^2\,\bar{\alpha}_k^2\,(1+(M-2)\,\epsilon_w)}. \tag{11}$$

Proof: We first use the fact that $|a+b| \geq \big||a|-|b|\big|$, as well as (9) to derive a lower bound on the numerator of (2):

$$\begin{aligned}\rho|\mathbf{w}_k^{\mathrm{H}}\mathbf{h}_k|^2 &\geq \rho\,\|\mathbf{h}_k\|^2\,(|a_k\,\mathbf{w}_k^{\mathrm{H}}\,\hat{\mathbf{h}}_k| - |\bar{a}_k\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|)^2 &\text{(12a)}\\ &\geq \rho\,\|\mathbf{h}_k\|^2\,(\alpha_k\,B - \bar{\alpha}_k\bar{B})^2, &\text{(12b)}\end{aligned}$$

[2] This choice is not unique if the channels do not form a base. In this case, additional constraints may be imposed, such as the maximization of the received power [1].

[3] The user may use different codebooks, which may improve system performance. However, as this would not impact the derivations in this paper, we will not consider this possibility.

which is equal to the numerator in (11) when $\rho = \frac{P}{M}$, that is, when M users are scheduled with equal power. Equality holds in (12a) when $a_k\, \mathbf{w}_k^{\mathrm{H}}\, \hat{\mathbf{h}}_k$ and $\bar{a}_k \mathbf{w}_k^{\mathrm{H}}\, \mathbf{e}_k$ have opposite phase. If this condition is not satisfied by the bound-achieving codebook vectors of the appendices, we can use orthogonal operations of reflections and rotations of these vectors to ensure the opposite signs of the terms of interest. Equality holds in (12b) when $|\mathbf{w}_k^{\mathrm{H}}\, \hat{\mathbf{h}}_k| = B$, which can happen because the bound in lemma 3 is tight. Thus, the bound derived in (12) is tight.

The upper bound on the denominator of (6) is a direct consequence of Lemma 2. Applying (8) to (6) and using $\rho = \frac{P}{M}$, the result follows. The overall bound is tight because the lemmas provide tight bounds. ■

Note that the tightness of the three lemmas and the theorem is shown by cumulatively constructing an example of channel and codebook vectors in which the bounds are achieved. As the same example is used in all cases, equality holds simultaneously for the three lemmas and the theorem. However, tightness is proved under the assumption that the number of scheduled users, M, is known. If fewer than M users are scheduled, the bound may be too pessimistic, as the system will present less interference. If more than M users are scheduled, the bound may no longer be valid, as the interference may be greater than that assumed in (11).

Finally, note that B and $\bar{B}$ depend on ϵ_h, which is assumed to be known by all users. Also, $\|\mathbf{h}_k\|$, α_k and $\bar{\alpha}_k$ can be computed at the user terminals. Thus, the bound is based solely on information available at the users. The BS, on the other hand, knows what beamforming vectors are actually used for transmission. In the sequel, we propose an adjustment to the lower bound that uses this information to improve the bound.

B. Lower Bound Adjustment at the Base Station

To motivate the need for adjusting the bound, consider a scenario of high quantization accuracy. In this case, $\alpha_k \to 1$ and $\bar{\alpha}_k \to 0$, so the the residual interference vanishes. As a result, when M users are scheduled,

$$\gamma_{LB_k} \to \frac{\frac{P}{M}\, \|\mathbf{h}_k\|^2\, B^2}{\sigma^2}. \tag{13}$$

Now, let $\gamma_{\text{Full CSIT}_k}$ be the SINR of the k-th user with full CSIT. In this case, there is no residual interference, so that we may write

$$\gamma_{\text{Full CSIT}_k} = \frac{\frac{P}{M}\, |\mathbf{w}_k^{\mathrm{H}} \mathbf{h}_k|^2}{\sigma^2}. \tag{14}$$

Since $B \leq |\mathbf{w}_k^{\mathrm{H}} \mathbf{h}_k|$, we see from (13) that, even with highly accurate quantization, γ_{LB_k} may not approach the SINR with full CSIT in (14). In this section, we propose an adjusted bound, $\hat{\gamma}_k$, that circumvents this problem.

We propose that the adjusted bound be obtained by multiplying γ_{LB_k} by a constant χ, so that $\hat{\gamma}_k = \chi\, \gamma_{LB_k}$. The value of χ should guarantee that $\hat{\gamma}_k$ is still a lower bound on the actual SINR. Now, the denominator of $\chi\, \gamma_{LB_k}$ is the same as that of γ_{LB_k}, which was proven to bound the denominator of γ_k. Thus, we only need to ensure that the numerator of $\chi\, \gamma_{LB_k}$ is a lower bound on the numerator of γ_k. As (12a) is a lower bound on this numerator, we see that $\chi\, \gamma_{LB_k} \leq \gamma_k$ as long as

$$\chi\, \rho\, \|\mathbf{h}_k\|^2\, (\alpha_k\, B - \bar{\alpha}_k\, \bar{B})^2 \leq \rho\, \|\mathbf{h}_k\|^2\, (\alpha_k\, \beta_k - \bar{\alpha}_k\, \bar{\beta}_k)^2, \tag{15}$$

where $\beta_k = |\mathbf{w}_k^{\mathrm{H}}\, \hat{\mathbf{h}}_k|$ and $\bar{\beta}_k = |\mathbf{w}_k^{\mathrm{H}}\, \mathbf{e}_k|$. From (15), we see that the optimal value for the adjustment is the largest value of χ that satisfies

$$\chi \leq f(\alpha_k, \beta_k, B) \triangleq \frac{(\alpha_k\, \beta_k - \bar{\alpha}_k\, \bar{\beta}_k)^2}{(\alpha_k\, B - \bar{\alpha}_k\, \bar{B})^2}. \tag{16}$$

However, the base station has no access to the value of α_k, thus it cannot compute $f(\alpha_k, \beta_k, B)$. Hence, the optimal value from the point of view of the BS, χ', is given by $\chi' = \min_{\alpha_k} f(\alpha_k, \beta_k, B)$. Now, note that $\bar{\alpha}_k < \alpha_k \leq 1$ and $\bar{\beta}_k < \beta_k \leq 1$. By differentiation, it is easy to see that, in this domain, $f(\alpha_k, \beta_k, B)$ is monotonically decreasing in α_k. Thus, its minimum is attained at $\alpha_k = 1$. Therefore we have that

$$\chi' = f(1, \beta_k, B) = \frac{\beta_k^2}{B^2}. \tag{17}$$

Note that the multiplication by χ' does not decrease the bound, since $\beta_k \geq B$. Also, with this adjustment, the bound approaches the true SINR as the accuracy of quantization increases.

IV. Numerical Results

In this section, we show numerical results that illustrate the bounds derived in this paper, and the improvement attained by the adjustment at the base station. To that end, we simulated a system with three transmit antennas, 100 users and a signal-to-noise ratio of $P/\sigma^2 = 20\,\text{dB}$. We used $\epsilon_h = 0.328$, obtained by trial and error to optimize the rates achieved by the proposed bound. Grassmannian codebooks [8] with nine bits were used for quantizing the CDI. We assume the CQI feedback to be unquantized. For each channel realization, we performed user selection using weighted clique search [7] based on the lower bound fed back by the users. In Fig. 1, we show the ratio between the two bounds derived in this paper and the actual SINR of the best scheduled user, computed with perfect channel knowledge. As expected, our bounds never exceed the actual SINR. We also point out that the adjustment at the base station always results in a higher value, providing a bound consistently larger (in this case, 11% on average) than the one computed by the user terminals.

We also compared the performance of the adjusted bound proposed in this paper to the CQI measure in [2], when both are used for rate adaptation. Fig. 2 shows the resulting sum rates as a function of the number of CDI bits for $N_t = 2, 3$ and 4 transmit antennas, for 100 users, $\epsilon_h = 0.328$, and $P/\sigma^2 = 10\,\text{dB}$. It can be seen that increasing the number of antennas can decrease the sum rate, depending on how many CDI bits are available. This is explained by the considerable decrease in quantization accuracy when complex dimensions are added to the channel vector and the number of bits is kept constant. The number of interferers is another factor that contributes to this behavior, as more users can be served simultaneously when more antennas are added. However, as seen in Fig. 2, for any number of CDI bits the proposed bound achieves the

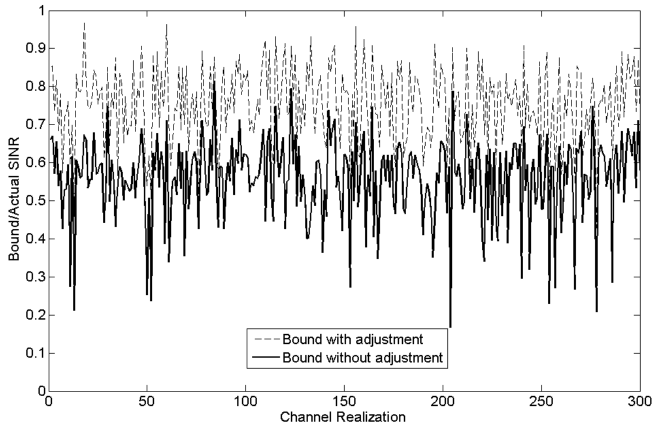

Fig. 1. Ratio between the bounds and the actual SINR.

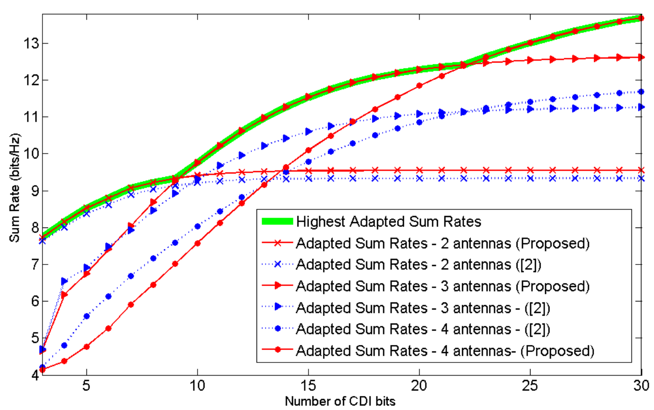

Fig. 2. Adapted sum rate as a function of CDI feedback load for $N_t = 2, 3$ and 4 transmit antennas.

best rates. Furthermore, as shown in [5], the rates achieved by the bounds proposed here are less sensitive to the choice of ϵ_h.

V. Conclusion

In this paper, we derived two bounds for the SINR in multiuser MISO systems with limited feedback that employ zero-forcing beamforming and user selection. The bounds assume that $M \leq N_t$ ϵ-orthogonal users are scheduled to transmit with equal power. The first bound is based on information available at the users, and is tight. The second bound can be computed at the base-station, and uses the knowledge of what the actual beamforming vectors are. As shown in the simulations, the adjusted bound improves the first bound. The performance of the bound for rate adaptation is also assessed through simulations.

Appendix A
Proof of Lemma 1

To prove that $|\mathbf{w}_i^{\mathrm{H}}\,\mathbf{w}_j| \leq \epsilon_w$ when $|\hat{\mathbf{h}}_i^{\mathrm{H}}\,\hat{\mathbf{h}}_j| \leq \epsilon_h$, let $\hat{\mathbf{H}} = \mathbf{QR}$ be the QR decomposition [9] of the matrix $\hat{\mathbf{H}}$ whose columns are the quantized channel vectors $\hat{\mathbf{h}}_k$ of the $M \leq N_t$ scheduled users. Note that the $M \times M$ matrix $\hat{\mathbf{H}}^{\mathrm{H}}\hat{\mathbf{H}} = \mathbf{R}^{\mathrm{H}}\mathbf{R}$ contains the inner products $\hat{\mathbf{h}}_i^{\mathrm{H}}\,\hat{\mathbf{h}}_j$, so its diagonal elements are equal to 1 and the magnitude of its off-diagonal elements are bounded by ϵ. We now exploit the upper-triangular structure of $\mathbf{R}$ to bound its elements. To that end, let κ_j be the lower bound on r_{jj}, and $\kappa_{i,j}$ be the upper bound on r_{ij}. Since $\|\hat{\mathbf{h}}_1\| = 1$, it is trivial that $|r_{11}| = \kappa_1 = 1$. For any $\hat{\mathbf{h}}_i$, $i > 1$, we have that $|\hat{\mathbf{h}}_i^{\mathrm{H}}\,\hat{\mathbf{h}}_1| \leq \epsilon_h$, and therefore $|r_{i1}| \leq \epsilon_h \triangleq \kappa_{i,1}$.

The lower bound on $|r_{22}|$, κ_2, is easily found by using the fact that $\|\hat{\mathbf{h}}_2\| = 1$, so that $|r_{22}|^2 = 1 - |r_{21}|^2$. Also, since $|r_{21}| \leq \epsilon_h$, we may write $|r_{22}|^2 \geq 1 - \epsilon_h^2 = \kappa_2^2$. For the terms $|r_{i2}|$, we use the fact that $|\hat{\mathbf{h}}_i^{\mathrm{H}}\,\hat{\mathbf{h}}_2| \leq \epsilon_h$ for all $i > 2$ to show that

$$|r_{i2}| \leq \frac{\epsilon_h + \epsilon_h^2}{\sqrt{1 - \epsilon_h^2}} = \kappa_{i,2}. \tag{18}$$

The same idea used for the components of $\hat{\mathbf{h}}_2$ can be applied to the general case of $\hat{\mathbf{h}}_i$ to find that

$$|r_{ij}| \leq \kappa_{i,j} = \frac{\epsilon_h + \sum_{l=1}^{j-1} \kappa_{i,l}^2}{\kappa_j} \quad \text{for } i > j \tag{19a}$$

$$|r_{jj}| \geq \kappa_j = \sqrt{1 - \sum_{l=1}^{j-1} \kappa_{i,l}^2}. \tag{19b}$$

From (19), we have the following relationships between the bounds:

$$\kappa_j^2 = \kappa_{j-1}^2 - \kappa_{j,j-1}^2, \tag{20}$$

$$\kappa_{i,j-1}^2 = \kappa_{i,j}\,\kappa_j - \kappa_{i,j-1}\,\kappa_{j-1}. \tag{21}$$

Now, the beamforming vectors can also be written in the basis $\mathbf{Q}$. From the ZFBF condition in (5), we see that

$$\mathbf{w}_k = w_{k,k}\,\mathbf{q}_k + w_{k,k+1}\,\mathbf{q}_{k+1} + \ldots + w_{k,M}\,\mathbf{q}_M, \tag{22}$$

so that $\mathbf{w}_k^{\mathrm{H}}\hat{\mathbf{h}}_j = 0$, for $j < k$. Since the order of the vectors is arbitrary, we can bound $|\mathbf{w}_i^{\mathrm{H}}\,\mathbf{w}_j|$ using any i and j. Thus, consider the inner product $|\mathbf{w}_M^{\mathrm{H}}\,\mathbf{w}_{M-1}|$. Now, since $\mathbf{w}_{M-1}$ is orthogonal to $\hat{\mathbf{h}}_M$, we have that $w_{M-1,M-1}^*\,r_{MM-1} = -w_{M-1,M}^*\,r_{MM}$. Recalling that $\|\mathbf{w}_{M-1}\| = 1$, so that $|w_{M-1,M-1}|^2 = 1 - |w_{M-1,M}|^2$, we may write

$$\frac{1 - |w_{M-1,M}|^2}{|w_{M-1,M}|^2} = \frac{|r_{MM}|^2}{|r_{MM-1}|^2}. \tag{23}$$

Now note that $|\mathbf{w}_M^{\mathrm{H}}\,\mathbf{w}_{M-1}| = |w_{M-1,M}|$, which is the desired bound. Let $\epsilon_w(M)$ be this bound. Applying (19), (20) and (21) to (23), we see, after some algebra, that

$$\epsilon_w(M+1) = \frac{\epsilon_w(M)}{1 - \epsilon_w(M)}. \tag{24}$$

Based on (24), we can use induction to prove the bound.

We now provide an example in which this bound is achieved, therefore proving that the bound is tight. Consider a matrix $\mathbf{A}$ in $\mathcal{C}^{M\times M}$, where $\mathbf{A}(i,i) = 1$ and $\mathbf{A}(i,j) = -\epsilon_h$ for $i \neq j$. Suppose that the Cholesky decomposition could be performed on $\mathbf{A}$ to yield $\mathbf{A} = \hat{\mathbf{H}}^{\mathrm{H}}\,\hat{\mathbf{H}}$, where $\hat{\mathbf{H}}$ is a possible realization of the quantized channel matrix. In this case, the Cholesky decomposition of $\mathbf{A}$ is such that equality holds for the bounds in (19). Thus, from (23), we conclude that the bound in lemma 1 is tight.

It remains to show that the Cholesky decomposition of $\mathbf{A}$ is possible, i.e., that $\mathbf{A}$ is a positive definite matrix [9]. To that end, we will use the Gershgorin circle theorem [9]. For the matrix $\mathbf{A}$ all circles are equal: centered at 1 and with a radius of $(M-1)\,\epsilon_h$. Since the matrix is Hermitian, its eigenvalues are real, and from the Gershgorin discs, they are limited to the interval $[1-(M-1)\,\epsilon_h; 1+(M-1)\,\epsilon_h]$. Note that the lower limit of this interval is in fact an eigenvalue with an associated eigenvector of $[1\,1\,\ldots\,1]^{\mathrm{T}}$. Therefore, $\mathbf{A}$ is definite positive if and only if $1-(M-1)\,\epsilon_h > 0$. Note that this condition fails when the bound in (7) is equal to trivial value of 1. In other words, when $\mathbf{A}$ is not positive definite, the bound is trivial.

Appendix B
Proof of Lemma 2

Here we shall derive an upper bound on the term $\sum_{j=1,j\neq k}^{M} |\mathbf{w}_j^{\mathrm{H}}\,\mathbf{e}_k|^2$. This term can be rewritten in matrix notation as

$$\sum_{j=1,j\neq k}^{M} |\mathbf{w}_j^{\mathrm{H}}\,\mathbf{e}_k|^2 = \mathbf{e}_k^{\mathrm{H}}\,\mathbf{W}_k\,\mathbf{W}_k^{\mathrm{H}}\,\mathbf{e}_k, \tag{25}$$

where the matrix $\mathbf{W}_k$, whose columns are the unit-norm beamforming vectors of all users but the k-th, is unknown, and $\mathbf{e}_k$ is a unit-norm vector. Now, the maximum value of the quadratic form in (25) is achieved when $\mathbf{e}_k$ is equal to the eigenvector associated with the largest eigenvalue of the matrix $\mathbf{W}_k\,\mathbf{W}_k^{\mathrm{H}}$. As this matrix is unknown, the problem seems ill-posed. However, we can determine the largest eigenvalue of matrices of the form $\mathbf{W}_k\,\mathbf{W}_k^{\mathrm{H}}$, subject to the known constraints that $\|\mathbf{w}_i\| = 1$ and $|\mathbf{w}_i^H\mathbf{w}_j| \leq \epsilon_w$.

Now, the non-zero eigenvalues of $\mathbf{W}_k\,\mathbf{W}_k^{\mathrm{H}}$ are equal to the non-zero eigenvalues of $\mathbf{W}_k^{\mathrm{H}}\,\mathbf{W}_k$ [9]. From Lemma 1, we know that the diagonal entries of matrix $\mathbf{W}_k^{\mathrm{H}}\,\mathbf{W}_k$ are equal to 1 and the absolute value of its off-diagonal entries is bounded above by ϵ_w. Thus, all the Gershgorin disks [9] of $\mathbf{W}_k^{\mathrm{H}}\,\mathbf{W}_k$ are centered in 1 and the largest radius possible is $(M-2)\,\epsilon_w$. Since the matrix is Hermitian, and therefore all its eigenvalues are real, the largest eigenvalue possible is $1+(M-2)\,\epsilon_w$. We conclude therefore that $\mathbf{e}_k^{\mathrm{H}}\,\mathbf{W}_k\,\mathbf{W}_k^{\mathrm{H}}\,\mathbf{e}_k \leq 1+(M-2)\,\epsilon_w$, as stated in Lemma 2.

To see that the bound is tight, simply note that $[1\,1\,\ldots\,1]^{\mathrm{T}}$ is an eigenvector of $\mathbf{W}_k^{\mathrm{H}}\,\mathbf{W}_k$ when $\mathbf{W}_k$ is the bound-achieving matrix in appendix A. The corresponding eigenvalue is precisely $1+(M-2)\,\epsilon_w$. Also, the channel is arbitrary, so that, given the bound-achieving codebook vectors in appendix A, we can find a channel realization that yields an error vector $\mathbf{e}_k$ that is proportional to $[1\,1\,\ldots\,1]^{\mathrm{T}}$. Thus, we conclude that there exist codebook vectors and a channel realization that achieve this bound.

Appendix C
Proof of Lemma 3

To prove (9), we begin by writing the quantized channel vectors as in appendix A. In this case, as $\mathbf{w}_M$ must be orthogonal to all channel vectors but $\hat{\mathbf{h}}_M$, and as $\|\mathbf{w}_M\| = 1$, then $\mathbf{w}_M = \mathbf{q}_M$. Thus, $\mathbf{w}_M^{\mathrm{H}}\hat{\mathbf{h}}_M = \hat{h}_{MM}$, so that $|\mathbf{w}_M^{\mathrm{H}}\hat{\mathbf{h}}_M| \geq \kappa_M$. As before, the order of the vectors is arbitrary, so that considering this particular inner product incurs no loss of generality. To determine κ_m, we divide both sides of (20) by κ_{M-1}, and we use (7) and some algebra to see that $\kappa_M^2 = \kappa_{M-1}^2 - (\epsilon_w(M)\kappa_{M-1})^2$. It can be seen that B in (10) satisfies this recursion. Now, the term $|\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|$ can be bounded by observing that $|\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|^2 + |\mathbf{w}_k^{\mathrm{H}}\,\hat{\mathbf{h}}_k|^2 \leq 1$, where equality holds if $\mathbf{w}_k$, $\hat{\mathbf{h}}_k$ and $\mathbf{e}_k$ are coplanar. Thus, defining $\bar{B}^2 \triangleq 1 - B^2$, we can use the fact that $|\mathbf{w}_M^{\mathrm{H}}\hat{\mathbf{h}}_M| \geq B$ to conclude that

$$|\mathbf{w}_k^{\mathrm{H}}\,\mathbf{e}_k|^2 \leq \bar{B}^2. \tag{26}$$

This proves the bound. To show that it is tight, note that the bound-achieving codebook vectors in appendix A and the channel realization in appendix B have $\kappa_M = B$. To ensure that $\mathbf{w}_k$, $\hat{\mathbf{h}}_k$ and $\mathbf{e}_k$ are coplanar, first fix $\hat{\mathbf{h}}_k$ and $\mathbf{e}_k$ as in appendices A and B. Now rotate the quantized channel matrix $\hat{\mathbf{H}}$ around the axis formed by $\hat{\mathbf{h}}_k$, so that $\hat{\mathbf{h}}_k$ remains constant. This would cause the ZFBF vector $\hat{\mathbf{w}}_k$ to rotate, without affecting inner products, and without changing $\hat{\mathbf{h}}_k$ and $\mathbf{e}_k$. Now, we just need to rotate $\hat{\mathbf{w}}_k$ until it gets to the desired plane.

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