The $\alpha - \eta - \mu$ and $\alpha - \kappa - \mu$ Fading Distributions

Gustavo Fraidenraich, Student Member, IEEE, and Michel Daoud Yacoub

Abstract—In this paper two new fading distributions, the $\alpha - \eta - \mu$ Distribution and $\alpha - \kappa - \mu$ Distribution, are presented. The $\alpha - \eta - \mu$ distribution includes the $\alpha - \mu$, Nakagami-$m$, Nakagami-$q$, Weibull, Hoyt, Rayleigh, Exponential, and the One-Sided Gaussian distributions as special cases. The $\alpha - \kappa - \mu$ distribution includes the $\kappa - \mu$, Nakagami-$m$, Weibull, Rice, Rayleigh, Exponential, and the One-Sided Gaussian distributions as special cases. Furthermore, it proposes estimators for the involved parameters and uses field measurements to validate the distributions.

Keywords—Fast fading distributions, Short-term fading, Rayleigh distribution, Rice distribution, One-sided Gaussian distribution, Hoyt distribution, Weibull distribution, Nakagami-$m$ distribution.

I. INTRODUCTION

A great number of distributions exist that well describe the statistics of the mobile radio signal. The long-term signal variation is known to follow the Lognormal distribution whereas the short term signal variation is described by several other distributions such as Hoyt, Rayleigh, Rice, Nakagami-$m$, and Weibull. It is generally accepted that the path strength at any delay is characterized by the short-term distributions over a spatial dimension of a few hundred wavelengths, and by the Lognormal distribution over areas whose dimension is much larger [1]. Among the short term fading distributions, Nakagami-$m$ has been given a special attention for its ease of manipulation and wide range of applicability. Although, in general, it has been found that the fading statistics of the mobile radio channel may well be characterized by Nakagami-$m$, situations are easily found for which other distributions such as Hoyt, Rice, and Weibull yield better results [1]–[4]. More importantly, situations are encountered for which no distributions seem to adequately fit experimental data, though one or another may yield a moderate fitting. Some researches [2] even question the use of the Nakagami-$m$ distribution because its tail does not seem to yield a good fitting to experimental data, better fitting being found around the mean or median. The well-known fading distributions have been derived assuming a homogeneous diffuse scattering field, resulting from randomly distributed point scatterers. The assumption of a homogeneous diffuse scattering field is certainly an approximation because the surfaces are spatially correlated characterizing a non-linear environment [5]. With the aim at exploring this non-homogeneity, two new fading distributions – the $\kappa - \mu$ Distribution and $\eta - \mu$ Distribution – have been presented in [6]–[8] and to explore the non-linearity of the propagation medium, which was addressed more recently in a new proposed general fading distribution, the $\alpha - \mu$ Distribution [9].

This paper presents two new fading distribution, the $\alpha - \eta - \mu$ and $\alpha - \kappa - \mu$ distributions. These distributions includes the $\alpha - \mu$, $\eta - \mu$, $\kappa - \mu$, Nakagami-$m$, Weibull, Rice, Hoyt, Rayleigh, and One-Sided Gaussian as special cases. In addition, the moments, the cumulative density function, and the estimators are derived for both distributions. Field measurements are used to validate the distributions.

II. THE $\alpha - \eta - \mu$ DISTRIBUTION

The $\alpha - \kappa - \mu$ distribution is a general fading distribution that can be used to represent the small-scale variation of the fading signal. For a fading signal with envelope $R$ and normalized envelope $P = \frac{R}{\sqrt{E[R^2]}}$, the $\alpha - \kappa - \mu$ probability density function is written as

$$f_P(p) = \frac{\alpha (\eta - 1)\frac{1}{\eta \mu} (\eta + 1)\frac{1}{\eta \mu} \rho (\frac{1}{\eta \mu} - 1)}{e^{\frac{1}{\eta \mu} + \frac{\rho}{\eta \mu} - 1} \sqrt{\pi \Gamma (\mu)}}$$

where $\alpha$ is the non-linearity parameter, $\eta \geq 0$, $I_\nu(\cdot)$ is the modified bessel function of the first kind, and $\Gamma(\cdot)$ is the Euler gamma function.

The $t - th$ moment $E[P^t]$ of $P$ can be attained as

$$E(P^t) = \frac{2^{\frac{1}{2} + \eta} \eta \frac{1}{\mu} (1 + \eta) (\frac{1}{\mu} + 2 \mu)}{\mu^\frac{1}{2} \Gamma (2\mu)}$$

$$\times 2F1 \left( \frac{t}{2} + \mu, \frac{t + \alpha}{2}, \frac{1}{2} + \mu, \frac{(-1 + \eta)^2}{(1 + \eta)^2} \right)$$

where $2F1$ is the Hypergeometric function [10].

In the same way, the cumulative probability function can be obtained in the usual integral manner, in series expansion or using the $Y_\mu(\cdot, \cdot)$ function presented in [8]

$$F_P(p) = \sum_{j=0}^{\infty} \frac{21-2j \sqrt{\pi (\eta - 1)^{2j} \eta^\mu (\eta + 1)^{-2j-2\mu}} j \Gamma (\mu) \Gamma (\frac{1}{2} + j + \mu)}{\mu \Gamma (2j) (2j + \mu)}$$

$$\times \Gamma (2 (j + \mu), \frac{(1 + \eta)^2}{2 \eta} \rho^2)$$

$$= 1 - Y_\mu \left( \frac{1 - \eta}{1 + \eta}, \frac{(1 + \eta)^2}{2 \eta} \sqrt{\frac{\mu}{\eta}} \rho^2 \right)$$

where $\Gamma(a, z) = \int_0^z t^{a-1}e^{-t}dt$ is the incomplete gamma function.
function and the function $Y_\mu(r, \cdot)$ is given in [8]

$$Y_\mu(a, b) = \frac{2^{1-\mu} (1 - a^2)^{\frac{\mu}{2}} \sqrt{\pi}}{\Gamma(\mu) a^\mu \frac{b}{2}} \times \int_b^\infty x^{2\mu} e^{-x^2} I_{\mu-\frac{1}{2}}(a x^2) \, dx \quad (5)$$

### III. THE $\alpha - \kappa - \mu$ DISTRIBUTION

The $\alpha - \kappa - \mu$ distribution is a general fading distribution that can be used to represent the small-scale variation of the fading signal. For a fading signal with envelope $R$ and normalized envelope $P$, the $\alpha - \kappa - \mu$ probability density function is written as

$$f_P(p) = \frac{\alpha \kappa^{\frac{1-\mu}{2}} (1 + \kappa) \Gamma(\mu, \mu \rho^2)}{e^{\mu \rho^2} (1 + \kappa)^{\frac{1}{2} \mu} \Gamma(\mu)} \times I_{\mu-1} \left(2 \sqrt{\kappa (1 + \kappa)} \mu \rho^2 \right) \quad (6)$$

where $\kappa > 0$.

The $t - th$ moment $E[P^t]$ of $P$ can be attained as

$$E(P^t) = \frac{\Gamma \left( \frac{t}{2} + \kappa \right)}{\Gamma(1 + \kappa)} \mu^{\frac{t}{2}} \mu \Gamma(\mu) \quad (7)$$

where $\Gamma$ is the Kummer confluent Hypergeometric function. In the same way, the cumulative probability function can be obtained in the usual integral manner, in series expansion or using the Marcum function [10], $Q_\mu(\cdot, \cdot)$. The expression for $F_P(p)$ can be found as

$$F_P(p) = \sum_{j=0}^\infty \frac{\kappa^j \mu^j \Gamma(j + \mu, (1 + \kappa) \mu \rho^2)}{j! e^{\mu \rho^2} \Gamma(j + \mu)} = 1 - Q_\mu \left(\sqrt{2 \mu \kappa}, \rho^2 \sqrt{2 \mu (1 + \kappa)} \right) \quad (8)$$

### IV. DERIVATION OF THE DISTRIBUTIONS

The envelope $R$, for the $\alpha - \eta - \mu$ distribution, and the $\alpha - \kappa - \mu$ distribution can be written as a function of the in-phase and quadrature elements of the multipath components so that

$$R^\alpha = \sum_{i=1}^n R_i^2 = \sum_{i=1}^n X_i^2 + Y_i^2 \quad (9)$$

where $X_i$ and $Y_i$ are independent Gaussian wide sense stationary processes in the in-phase and quadrature components of the propagated wave, respectively. For the $\alpha - \eta - \mu$ distribution assume that $E(X_i) = 0$, $E(Y_i) = 0$, $\text{Var}(X_i) = \sigma_X^2$, and $\text{Var}(Y_i) = \sigma_Y^2$, where $E(\cdot)$ is the mean operator and $\text{Var}(\cdot)$ is variance operator. For the $\alpha - \kappa - \mu$ distribution assume that $E(X_i) = \mu_X$, $E(Y_i) = \mu_Y$, $\text{Var}(X_i) = \sigma^2$, and $\text{Var}(Y_i) = \sigma^2$. Now we define the following parameters $\eta = \frac{\sigma_Y^2}{\sigma_X^2}$, as the ratio between the in-phase scattered wave and the quadrature scattered wave, and $\kappa = \frac{\sigma^2}{\sigma^2} = \frac{\sigma^2}{\sigma^2}$, as the ratio between the in-phase dominant component and the quadrature dominant component. Departing from (10) the joint gaussian probability density function $f_{X_i, Y_i}(x_i, y_i)$ can be found. Using $f_{R_i, \theta_i}(r_i, \theta_i) = r_i f_{X_i, Y_i}(r_i \cos(\theta_i), r_i \sin(\theta_i))$ to found $f_{R_i, \theta_i}(r_i, \theta_i)$ and by means of an integration over $\theta_i$, $f_{R_i}(r_i)$ is achieved. Making the following transformation of variables $W = R^\alpha$ and $R = \sum_{i=1}^n W_i$, hence $W = \sum_{i=1}^n W_i$. Performing the Laplace transform $\mathcal{L}[f_{W_i}(w_i)]$ and knowing that $W_i, i = 1, 2, \cdots, n$ are mutually independent, the Laplace transform of $\mathcal{L}[f_{W_i}(w_i)]$ is found as a $n$-fold multiplication of $\mathcal{L}[f_{W_i}(w_i)]$. Applying the inverse transformation as $R = (W)^{1/\alpha}$, and $P = \frac{R}{\sqrt{R^2 + \sigma^2}}$, (1) and (6) follow directly.

For the $\alpha - \eta - \mu$, $n/2 = \mu$ ($\mu$ being the real extension of $n/2$), and $E[R^\alpha] = \eta (1 + \eta) \sigma^2$. In the same way, for the $\alpha - \kappa - \mu$, $n = \mu$ ($\mu$ being the real extension of $n$), and $E[R^\alpha] = (1 + \kappa)2 \mu \sigma^2$.

### V. ESTIMATORS

The estimation of the corresponding parameters of the distributions can be performed using the generalized inverse of the normalized variance given as [9]

$$\beta \mu = \frac{E^2(P^\beta)}{E(P^2) - E^2(P)} \quad (11)$$

where $E(R^\beta)$ can be obtained from (2) and (7). Note that the Nakagami $m$ parameter [11] can be obtained from (11) using $m = \alpha \mu$.

For the $\alpha - \eta - \mu$, (11) is given as

$$\beta \mu = \frac{1 - \frac{(1 + \eta)2 \mu}{\Gamma(2 \mu)} \mu \Gamma(\frac{2 \beta}{\alpha} + 2 \mu)}{2^{2 \mu} \eta \mu \Gamma(\frac{\beta \alpha}{\alpha} + 2 \mu) \times 2 F_1 \left(\frac{\beta}{\alpha} + \mu, \frac{1}{2} \left(1 + \frac{2 \beta}{\alpha}\right) + \mu, \frac{1}{2} + \mu, \frac{(1 + \eta)2 \mu}{(1 + \eta)2 \mu} \right)^{-1} \times 2 F_1 \left(\frac{\beta}{\alpha} + \mu, \frac{1 + \beta}{\alpha} + \mu, \frac{1}{2} + \mu, \frac{(1 + \eta)2 \mu}{(1 + \eta)2 \mu} \right) \quad (12)$$

in the same way, for the $\alpha - \kappa - \mu$, (11) is given as

$$\beta \mu = \frac{\left( e^{\mu \rho^2} \Gamma(\frac{2 \beta}{\alpha} + 2 \mu) \mu \Gamma(\frac{\beta \alpha}{\alpha} + 2 \mu) \right)^{-1}}{\Gamma(\frac{\beta}{\alpha} + \mu) \times 1 F_1 \left(\frac{\beta}{\alpha} + \mu, \frac{1}{2} \left(1 + \frac{2 \beta}{\alpha}\right) + \mu, \frac{1}{2} + \mu, \frac{(1 + \eta)2 \mu}{(1 + \eta)2 \mu} \right) \quad (13)}$$

Note that for both the $\alpha - \eta - \mu$ distribution and the $\alpha - \kappa - \mu$ distribution, it is necessary three equations to find out the three corresponding parameters, namely $\alpha$, $\eta$, or $\kappa$, and $\mu$. For this end, we choose three distinct values for $\beta$ in (11), for example, $\beta = 1, 1.5, 2$, and then using a non-linear numerical method it is possible to find a possible solution to the three parameters.

Although exact, this method for estimation of the parameters suffers from non-stability of the roots. Thus for some cases it may be possible to find more than one possible solution for the corresponding parameters.

#### A. The $\alpha - \eta - \mu$ distribution and the $\alpha - \eta - \mu$ distribution

In the plane using logarithmic scale (fading plane), given a fixed $\alpha$ and $\mu$ parameters, the curves belonging to the $\alpha - \kappa - \mu$ regime are all found above the $\alpha - \mu$ curve whereas those belonging to the $\alpha - \eta - \mu$ distribution are all encountered below the $\alpha - \mu$ curve. Generally speaking, the $\alpha - \mu$ distribution can be thought of as a mean distribution,
which divides the fading plane into two: the upper plane, described by the \( \alpha - \kappa - \mu \) Distribution, and the lower plane, described by the \( \alpha - \eta - \mu \) Distribution. In Fig. 1, such a feature is illustrated. This is a very interesting attribute that can be used in order to choose the best distribution to fit experimental data, as explained next. For a given set of data, the \( \alpha \) and \( \mu \) parameters are calculated, as reported in [9], and the experimental data are plotted in the fading plane. In case these data are found above the \( \alpha - \mu \) curve, then the best distribution to fit these data is the \( \alpha - \kappa - \mu \) Distribution; otherwise, the best distribution is the \( \alpha - \eta - \mu \) Distribution. Note that the versatility provided by the use of two parameters renders these two distributions suited for applications in which other distributions fail to yield good fitting, particularly for low values of the fading envelope. Moreover, for a fixed \( \alpha \mu \) and varying \( \alpha \) the curve moves up or down according to the \( \mu \) parameter. This behavior is illustrated in Fig. 2 for \( \alpha = 1.25 \).

VI. THE \( \alpha - \eta - \mu \), \( \alpha - \kappa - \mu \), AND THE OTHER FADING DISTRIBUTIONS

The \( \alpha - \eta - \mu \) Distribution is a general fading distribution that includes the Rayleigh, Hoyt, Nakagami-\( m \), \( \eta - \mu \), Weibull, and One-Sided Gaussian distributions as special cases. The \( \eta - \mu \) distribution can be obtained from the \( \alpha - \eta - \mu \) Distribution in an exact manner by setting \( \alpha = 2 \). From the \( \eta - \mu \) distribution, the Nakagami-\( m \) curve is obtained for \( \eta \to 1 \), in which case \( \mu = m/2 \), or, equivalently, for \( \eta \to 0 \), in which case, \( \mu = m \). The Hoyt distribution can be obtained from the \( \eta - \mu \) Distribution by setting \( \mu = 1 \) and using the relation \( b = \frac{1+\alpha}{2} \), where \( b \) is the Hoyt parameter. From the Hoyt distribution the One-Sided Gaussian is obtained for \( b \to \pm 1 \) (\( \eta \to 0 \) or \( \eta \to \infty \)). In the same way, from the Hoyt distribution the Rayleigh distribution is obtained in an exact manner for \( b = 0 \) (\( \eta = 1 \)). Finally, the Weibull distribution can be obtained from the \( \alpha - \eta - \mu \) Distribution by setting \( \eta = 1 \) and \( \mu = 1 \). Table I shows how these distributions can be obtained from the \( \alpha - \eta - \mu \) Distribution. In this table, the symbols \( m \), \( b \), and \( k \) are the Nakagami, Hoyt, and Rice parameters, respectively.

The \( \alpha - \kappa - \mu \) Distribution is a general fading distribution that includes the Rayleigh, Rice, Nakagami-\( m \), \( \kappa - \mu \), Weibull, and One-Sided Gaussian distributions as special cases. The \( \kappa - \mu \) distribution can be obtained from the \( \alpha - \eta - \mu \) Distribution in an exact manner by setting \( \alpha = 2 \). The Nakagami-\( m \) distribution can be obtained from the \( \kappa - \mu \) distribution by setting \( \kappa = 0 \). The Rice distribution can obtained from the \( \kappa - \mu \) distribution by setting \( \mu = 1 \) and using the relation \( \kappa = k \), where \( k \) is the Rice parameter. In the same way, from the Rice distribution the Rayleigh distribution is obtained in an exact manner for \( k = 0 \). Finally, the Weibull distribution can be obtained from the \( \alpha - \kappa - \mu \) distribution by setting \( \kappa = 0 \) and \( \mu = 1 \). Table II shows how these distributions can be obtained from the \( \alpha - \kappa - \mu \) Distribution.

VII. VALIDATION THROUGH FIELD MEASUREMENTS

A series of field trials was conducted at the University of Campinas (Unicamp), Brazil, in order to investigate the short term statistics of the fading signal. The reception setup comprised a vertically polarized omnidirectional antenna, a low noise amplifier, a spectrum analyzer, a data acquisition apparatus, a notebook computer, and a distance transducer [12]. The transmission consisted of a CW tone at 1.8 GHz with transmitter and receiver placed within buildings (indoor propagation). The spectrum analyzer was set to zero span and centered at the desired frequency, and its video output used as the input of the data acquisition and processing equipment. The local mean was estimated through the moving average method, the average being conveniently taken over samples symmetrically adjacent to every point, a procedure widely reported in the literature [13].

Using the method reported in Sec. V, the parameters were estimated and used to fit the experimental data. Fig. 3 shows some sample plots illustrating the adjustment obtained by the \( \alpha - \eta - \mu \) Distribution as compared to the Rayleigh, Nakagami-\( m \), Weibull, and Hoyt ones. In Fig. 3, we note that the Weibull fitting is not adequate at the tail of the distribution but it yields a good fitting for higher values of the envelope. In contrast, the \( \alpha - \kappa - \mu \) distribution provides a very good fitting for the lower values of the envelope and tends to keep track of the change of concavity of the true curve. Fig. 4 shows some sample plots illustrating the adjustment obtained by the \( \alpha - \kappa - \mu \) Distribution as compared to the Rayleigh, Nakagami, Weibull, and Rice ones. Note how the \( \alpha - \kappa - \mu \) distribution tends to keep track of the change of concavity of the true curve. In contrast with the others that have a monomodal behavior.

A numerical measure of the mean error deviation\(^1\) between the true curve and the distributions chosen to fit the experimental data (namely, Weibull, Rayleigh, Hoyt, Nakagami-\( m \), \( \alpha - \eta - \mu \), and \( \alpha - \kappa - \mu \)) has been calculated for all of the cases. Table III shows these for the curves in Figs. 3 and 4. In all of them, the error for the \( \alpha - \eta - \mu \) and \( \alpha - \kappa - \mu \) distributions is lower than that for the other distributions, although, the Weibull distribution also gives good results. On the other hand, as opposed to the Weibull one, the \( \alpha - \eta - \mu \) and \( \alpha - \kappa - \mu \) Distributions tend to reproduce or very closely follow the trends (concavity and/or convexity) of the true curve.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \epsilon ) (Fig. 3)</th>
<th>( \epsilon ) (Fig. 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha - \eta - \mu )</td>
<td>0.299</td>
<td>0.195</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>1.280</td>
<td>3.718</td>
</tr>
<tr>
<td>Weibull</td>
<td>0.659</td>
<td>0.523</td>
</tr>
<tr>
<td>Nakagami-( m )</td>
<td>1.731</td>
<td>0.744</td>
</tr>
<tr>
<td>Rice</td>
<td>—</td>
<td>0.574</td>
</tr>
<tr>
<td>Hoyt</td>
<td>1.32</td>
<td>—</td>
</tr>
</tbody>
</table>

Table III - Mean error deviation for Figs. 3 and 4.

VIII. CONCLUSIONS

This paper presented two new fading distributions, the \( \alpha - \eta - \mu \) and the \( \alpha - \kappa - \mu \) distribution. These distributions include the \( \alpha - \mu \), \( \kappa - \mu \), \( \eta - \mu \), Nakagami-\( m \), Nakagami-\( q \), Weibull,

\(^1\)The mean error deviation between the measured data \( x_i \) and the theoretical value \( y_i \) (Weibull, Rayleigh, Hoyt, Rice, Nakagami-\( m \), \( \alpha - \eta - \mu \) or \( \alpha - \kappa - \mu \)) is defined as \( \epsilon = \frac{1}{N} \sum_{i=1}^{N} \frac{|y_i - x_i|}{x_i} \), where \( N \) is the number of points.
Hoyt, Rice, Rayleigh, and One-Sided Gaussian distributions, as special cases. Furthermore, it proposed estimators for the involved parameters and used field measurements to validate the distributions.

**Table I - The \( \alpha - \eta - \mu \) distribution and the other fading distributions.**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \alpha )</th>
<th>( \eta )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Hoyt</td>
<td>( \alpha )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>One-Sided Gaussian</td>
<td>2</td>
<td>( \eta )</td>
<td>( \mu )</td>
</tr>
<tr>
<td>Weibull</td>
<td>( \alpha )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Nakagami-( m )</td>
<td>( \alpha )</td>
<td>1</td>
<td>( \mu )</td>
</tr>
</tbody>
</table>

**Table II - The \( \alpha - \kappa - \mu \) distribution and the other fading distributions.**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( \alpha )</th>
<th>( \kappa )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rayleigh</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Rice</td>
<td>( \alpha )</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>Rice</td>
<td>( \kappa )</td>
<td>0</td>
<td>( \mu )</td>
</tr>
<tr>
<td>One-Sided Gaussian</td>
<td>2</td>
<td>0</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>Weibull</td>
<td>( \alpha )</td>
<td>0</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>Nakagami-( m )</td>
<td>( \alpha )</td>
<td>0</td>
<td>( \kappa )</td>
</tr>
<tr>
<td>Nakagami-( m )</td>
<td>( \alpha )</td>
<td>0</td>
<td>( \mu )</td>
</tr>
</tbody>
</table>

Fig. 1. The \( \alpha - \mu \), \( \alpha - \eta - \mu \), and the \( \alpha - \kappa - \mu \) distributions

Fig. 2. The \( \alpha - \mu \), \( \alpha - \eta - \mu \), and the \( \alpha - \kappa - \mu \) distributions

Fig. 3. The \( \alpha - \eta - \mu \) distribution function adjusted to data of an indoor propagation measurement experiment at 1.8 GHz conducted at Unicamp.

**REFERÊNCIAS**

[8] M. D. Yacoub. The \( \kappa - \mu \) Distribution and the \( \eta - \mu \) Distribution. *Accepted for publication in IEEE Trans. on Ant. and Propag.*, 2006.
Fig. 4. The $\alpha - \kappa - \mu$ distribution function adjusted to data of an indoor propagation measurement experiment at 1.8 GHz conducted at Unicamp.


